CYCLOTRON ABSORPTION OF WAVES IN PLASMA PROPAGATING ACROSS A NONUNIFORM MAGNETIC FIELD

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1. Introduction

The damping of electromagnetic waves propagating strictly across the uniform magnetic field is known to vanish [1]. The inclusion of non-local effects related to electron motion along the magnetic field non-uniform along the magnetic lines of force (m.l.f) leaves this result unchanged [2]. In the non-uniform magnetic field with the transverse gradient and straight m.l.f. the local non-uniformity of the rate at which resonant particle rotate along the Larmor orbit leads to the local resonance [3]

$$\omega = n\omega_{Ci}(x) + nv_y / L_B \text{ , } (n=1,2,3...) \tag{1}$$

where $\omega_{Ci} = \frac{eB_0}{m_i c}\left(1+\frac{x}{L_B}\right)$, L_B is the characteristic length of the magnetic field change, the magnetic axis is directed along z-axis, x-axis is directed along the magnetic field non-uniformity.

If $k_\perp \rho_i << 1$, then one can neglect the Doppler shift in the cyclotron resonance condition, $\rho_i = v_{Ti}/\omega_{Ci}$ is the particle Larmor radius, $v_{Ti} = \sqrt{T_i/m_i}$.

In this report we are studying the cyclotron resonance for fast magnetosonic waves (FMSW) in a small pressure plasma ($\beta_{e,i} = 4\pi n_0 T_{e,i}/B^2 << 1$) including multiple resonances $n \geq 2$ as well as the fundamental resonance (*n=1*).

2. Multiple cyclotron resonance.

The dispersion relation for FMSW propagating across the magnetic field has the form

$$\varepsilon_{11}N^2 = \varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21} . \tag{2}$$

To the order of magnitude

$$N^2 = N_A^2 = \frac{c^2}{v_A^2} = \sum_i \frac{\omega_{pi}^2}{\omega_{Ci}^2} . \tag{3}$$

In the next approximation accounting for the small cyclotron damping one can put

$$\begin{aligned} &\varepsilon_{11} = \varepsilon_1 + \delta\varepsilon_{11}, \qquad \varepsilon_{22} = \varepsilon_1 + \delta\varepsilon_{22}, \\ &\varepsilon_{12} = \varepsilon_2 + i\delta\varepsilon_{11}, \qquad \varepsilon_{21} = -\varepsilon_2 - i\delta\varepsilon_{22}, \\ &\varepsilon_1 = -\frac{\omega_{pi}^2}{\omega^2 - \omega_{Ci}^2}, \quad \varepsilon_2 = -i\frac{\omega_{pi}^2\omega}{\omega_{Ci}\left(\omega^2 - \omega_{Ci}^2\right)}, \end{aligned} \tag{4}$$

$$\delta\varepsilon_{11} = -(i)^n \sqrt{\pi}\,\frac{\omega_{pi}^2}{\omega}\frac{(k\rho_i)^{n-1}}{2^{3n/2}}\frac{L_B}{n!\,v_{Ti}}\exp\left(-\zeta^2 + ix_i\sqrt{2}\zeta\right)\left[\zeta H_n(-\zeta) + \frac{1}{2}H_{n+1}(-\zeta)\right],$$

$$\delta\varepsilon_{11} = -(i)^n \sqrt{\pi}\,\frac{\omega_{pi}^2}{\omega}\frac{(k\rho_i)^{n-1}}{2^{3n/2}}\frac{L_B}{n!\,v_{Ti}}\exp\left(-\zeta^2 + ix_i\sqrt{2}\zeta\right)\zeta H_n(-\zeta).$$

Regarding the elements $\delta\varepsilon_{11}, \delta\varepsilon_{22}$ as small we obtain from eq. (2) that the imaginary part of the refractive index $\kappa = \operatorname{Im} N$ for one ion species and waves propagating along x-axis is equal to

$$\kappa = \sqrt{\pi}\,\frac{n-1}{n}\frac{(k\rho_i)^{n-1}}{2^{3n/2+1}n!}\frac{L_B\omega_{Ci}}{v_{Ti}}\exp\left(-\zeta^2\right)\left[(n-1)\zeta H_n(-\zeta) + \frac{n}{2}H_{n+1}(-\zeta)\right]\times$$
$$\times \operatorname{Re}\left(i^{n+1}\exp\left(ix_i\sqrt{2}\zeta\right)\right) \tag{5}$$

where $\zeta(x) = \dfrac{[\omega - n\omega_{Ci}(x)]|L_B|}{\sqrt{2}nv_{Ti}}$, $H_n(\zeta)$ is the Hermite polynomial.

One proves easily that for the resonance *n=2m (m=1,2,3...)* the term $ik\rho_i\zeta$ of the expansion $\exp(ik\rho_i\zeta)$ makes the main contribution to (5). In this case for *n=2* expression (5) is always positive and it predicts the local damping of waves in the range $\omega > \omega_{Ci}$ as well as in the range $\omega < \omega_{Ci}$. The expression for the optical thickness $\tau = \int_{-\infty}^{\infty} \operatorname{Im}\kappa dx$ coincides with those obtained with the inclusion of other mechanisms of cyclotron absorption: due to longitudinal Doppler effect or binary collisions. But already at *n=4* one can prove that the quantity κ may be positive as well as negative meaning the local instability of the FSMW. But for the waves passing through the resonance zone the optical thickness happens to vanish: absorption and damping of FMSW cancel. Such cancellation occurs when one expands the term $\exp(ik\rho_i\zeta)$ up to the terms proportional to $(k\rho_i\zeta)^{2m-3}$. Only keeping terms with $(k\rho_i\zeta)^{2m-1}$ yields the final positive expression for κ

$$\frac{\operatorname{Im} k}{k} = \sqrt{\pi}\,\frac{n-1}{n}\frac{(k\rho_i)^{n-1}}{2^{3n/2+1}n!}\frac{L_B\omega_{Ci}}{v_{Ti}}\exp\left(-\zeta^2\right)\left[(n-1)\zeta H_n(-\zeta) + \frac{n}{2}H_{n+1}(-\zeta)\right]\times$$
$$\times \operatorname{Re}\left(i^{n+1}\exp\left(ix_i\sqrt{2}\zeta\right)\right) \tag{6}$$

For even numbers *n=2m* one should put *s=2m-1* in formula (6) and for odd numbers *n=2m+1* one should take the term with *s=2m*. The calculation of the optical thickness due to damping (6) yields

$$\tau = \frac{\pi}{2}\frac{(n-1)^2}{2^n n!}\left(\frac{v_{Ti}n}{v_A}\right)^{2n-2}\frac{\omega L_B}{v_A} \tag{7}$$

This result has been obtained for the resonance *n=2* in Ref. [3] . The same expression is obtained if one calculates the cyclotron damping for FMSW propagating almost across the magnetic field $\left(k_\parallel << k_A \cong \omega/v_A\right)$ and recalls [1] that

$$\frac{\operatorname{Im} k}{k} = \sqrt{\frac{\pi}{8}}\frac{(n-1)^2}{2^n n!}\left(\frac{v_{Ti}}{v_A}n\right)^{2n-2}\frac{\omega}{k_\parallel v_{Ti}}\exp\left[-\left(\frac{\omega - n\omega_{Ci}}{\sqrt{2}k_\parallel v_{Ti}}\right)^2\right]. \tag{8}$$

The same expression (7) is also obtained when one calculates the collisional absorption at $\omega \approx n\omega_{Ci}$:

$$\frac{\operatorname{Im} k}{k} = \frac{(n-1)^2}{2^{n+1} n!}\left(\frac{v_{Ti}}{v_A} n\right)^{2n-2} \frac{\omega\nu}{\left(\omega - n\omega_{Ci}\right)^2 + \nu^2}, \tag{9}$$

where ν is the collision rate.

3. Fundamental resonance.

At $\omega \approx \omega_{Ci}$, the expressions for the dielectric tensor ε_{ij} with $k\rho_i << 1$ are reduced to the following

$$\varepsilon_{11} = \varepsilon_{1T} - N_A^2/4, \quad \varepsilon_{22} = \varepsilon_{2T} - N_A^2/4,$$
$$\varepsilon_{12} = i\varepsilon_{1T} - 3i\, N_A^2/4, \varepsilon_{21} = -i\varepsilon_{2T} + 3i\, N_A^2/4, \tag{10}$$

where

$$\varepsilon_{1T} = i\frac{N_A^2}{4}\frac{L_B\omega_{Ci}}{v_{Ti}}\left\{\sqrt{2\pi}W\left(\zeta - \frac{ix_i}{\sqrt{2}}\right) + 2ix_i\left[1 + i\sqrt{\pi}\zeta W\left(\zeta - \frac{ix_i}{\sqrt{2}}\right)\right]\right\},$$

$$\varepsilon_{1T} = \sqrt{2}\frac{N_A^2}{2}\frac{L_B\omega_{Ci}}{v_{Ti}}\left\{(\zeta + x_i)\left[1 + i\sqrt{\pi}\zeta W\left(\zeta - \frac{ix_i}{\sqrt{2}}\right)\right] - \frac{ix_i}{\sqrt{2}}\right\}. \tag{11}$$

Here

$$W(\zeta) = \exp\left(-\zeta^2\right)\left[1 + \left(2i/\sqrt{\pi}\right)\int_0^{\zeta}\exp\left(t^2\right)dt\right], \tag{12}$$

$$\zeta = \frac{\left[\omega - \omega_{Ci}(x)\right]L_B}{\sqrt{2}v_{Ti}}, \quad x_i = k\rho_i .$$

Neglecting in (11) the terms containing $x_i << 1$ yields the estimate $\varepsilon_{1T} \sim \varepsilon_{2T} \sim N_A^2 \dfrac{L_B}{\rho_i}$. Terms in (11) containing *x* are equal to the order of magnitude to $N_A^2 k_A L_B$. These terms are still considerably larger than the non-resonant terms in (10) which are equal to N_A^2 to the order of magnitude because our consideration assumes the applicability of the WKB method $\left(k_A L_B >> 1\right)$.

Taking into account these inequalities and keeping in mind that the largest terms in eq.(2) cancel yield

$$\frac{N^2}{N_A^2} - \frac{1+2\zeta^2}{2} - x_i\sqrt{2}\zeta\frac{N^2}{N_A^2} - x_i\zeta\frac{\sqrt{2}-1}{\sqrt{2}} =$$
$$= -\frac{i}{\sqrt{2\pi}W\left(\zeta - ix_i/\sqrt{2}\right)}\left[\sqrt{2}\zeta + 2x_i\frac{N^2}{N_A^2} + \left(\sqrt{2}-1\right)x_i\right] \tag{13}$$

Neglecting in (13) the small terms proportional to *x*, we obtain that $N = N_R + iN_I$, where $N(\zeta)$ is determined from the equation

$$\frac{N^2(\zeta)}{N_A^2} = \frac{1}{2} + \zeta^2 - \frac{i\zeta}{\sqrt{\pi}W(\zeta)}. \tag{14}$$

In particular cases we have

$$\frac{N_R}{N_A} = \frac{1}{\sqrt{2}}, \qquad \frac{N_I}{N_A} = -\frac{\zeta}{\sqrt{2\pi}}, \qquad \left(|\zeta| << 1\right),$$

$$\frac{N_R}{N_A} = 1, \qquad \frac{N_I}{N_A} = -\frac{\sqrt{\pi}}{2}\zeta^3 \exp\left(-\zeta^2\right), \qquad \left(|\zeta| >> 1\right), \tag{15}$$

The imaginary part of the refractive index is the odd function of ζ. At $\zeta > 0$ we would have the FMSW amplification in our consideration whereas at $\zeta < 0$ the FMSW would be absorbed. Therefore the optical thickness vanishes in this approximation.

The real part of the refractive index is the even function and it changes essentially within the interval $|\zeta| \sim 1$, that is $x \sim \rho_i$. As we assume that the plasma density changes over the characteristic length $L_n << L_B \left(\beta_i << 1\right)$, then the WKB approximation loses its validity for the fundamental resonance $\omega \approx \omega_{Ci}$. In the case considered it is better to use the narrow-layer approximation when the coefficients in Maxwell's equations that contain $W(\zeta)$ experience strong changes in the layer where $|\zeta| \le 1$. However, to make sure that under the resonance $\omega = \omega_{Ci}$ the cyclotron damping still exists with the transverse gradient taken into account, we keep in equation (13) the terms order of *x*. Then for the additional damping rate we get the estimate even with respect to the sign of $\omega - \omega_{Ci}$

$$\frac{\mathrm{Im}\, k}{k_A} \sim \frac{v_{Ti}}{v_A}, \qquad \left(\frac{|\omega - \omega_C| L_B}{v_{Ti}} \le 1\right). \tag{16}$$

Note that due to the inapplicability of the plane wave approximation the magnitude of the numerical factor and even the sign of *ImN* in (16) remain indefinite.

4. Conclusion

The absorption coefficients for FMSW propagating across the magnetic field under multiple cyclotron resonance conditions $\omega \approx n\omega_{Ci}$ obtained in the gyrokinetic approximation lead to the optical thickness coinciding with one formed by the Doppler spread of the cyclotron resonance or by collisions.

The fundamental resonance $\omega \approx \omega_{Ci}$ cannot be considered in the local approximation as the WKB technique fails but it can be considered in the narrow-layer approximation [4].

References

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