ON THE WAVE SCATTERING IN PLASMA WITH SMALL SCALE DENSITY IRREGULARITIES

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A systematic approach to the problem of wave scattering on plasma density irregularities is developed. The scattering process is treated as a wave field spatial evolution while crossing the scattering region. We show that the amplitude modulation and apparent spectral broadening of VLF transmitter signals are general consequences of the wave scattering.

We consider a magnetized plasma, with the ambient magnetic field $\vec{B}_0$ directed along the z-axis, and study the wave problem in the case when a limited in the z-direction region of plasma density irregularities is in the way of the wave propagation. In this region, the wave scattering on plasma density irregularities takes place. We look for the wave field outside the scattering region.

The electromagnetic field of a monochromatic wave is expressed through the real part of complex vector potential as follows:

$$\vec{\mathcal{B}} = \mathrm{Re}\{\mathrm{rot}\vec{\mathcal{A}}\} \ , \quad \vec{\mathcal{E}} = - - \mathrm{Re}\{\frac{\partial \vec{\mathcal{A}}}{\partial t}\} \ ; \quad \mathcal{A}_j = A_j(x,z)e^{-i\omega t} \ , \quad (j = x,y,z) \ . \qquad (1)$$

In the cold plasma approximation, the components $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of the dielectric tensor for low frequency whistler mode waves are reduced to (in standard notation, e.g. [1]):

$$\frac{\varepsilon_1}{c^2} = \frac{\omega_p^2 + \omega_c^2}{c^2\omega_c^2} - \frac{\omega_p^2 + \omega_c^2}{c^2\omega_c^2}\frac{\omega_{LH}^2}{\omega^2} \equiv \alpha - \frac{\beta}{\omega^2} \ ;$$
$$\frac{\varepsilon_2}{c^2} = - - \frac{\omega_p^2}{c^2\omega\omega_c} \equiv - - \frac{\gamma}{\omega} \ ; \quad \frac{\varepsilon_3}{c^2} = - - \frac{\omega_p^2}{c^2\omega^2} \equiv - - \frac{\eta}{\omega^2} \ , \qquad (2)$$

where $\omega_p, \ \omega_c$ are the electron plasma and cyclotron frequencies, respectively, and ω_{LH} is the lower hybrid resonance frequency. The expressions for the introduced quantities α, β, γ and η are evident from their definition. With the above introduced quantities, Maxwell equations give (see [2] for detais]):

$$\frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_z}{\partial x \partial z} = -\omega^2\alpha A_x + \beta A_x + i\omega\gamma A_y \ ;$$
$$\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} = - - i\omega\gamma A_x \ ; \quad \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_x}{\partial x \partial z} = \eta A_z \ . \qquad (3)$$

Equations (3) are written in two dimensional geometry, i.e. all quantities are supposed to depend only on the x and z coordinates.

The parameters α, β, γ and η introduced above (see (2)) are represented in the form of constant part plus a perturbation describing plasma irregularities, i.e.

$$\alpha(x,z) = \alpha_0 + \tilde{\alpha}(x,z) \; ; \qquad \tilde{\alpha} \ll \alpha_0 \; . \tag{4}$$

Similar expressions and inequalities are assumed for β, γ and η .

The set of equations (3) can be written in a matrix form as follows:

$$\widehat{M}_{ij} A_j = \widetilde{M}_{ij} A_j \; , \tag{5}$$

where $\widehat{M}_{ij}$ is a matrix differential operator corresponding to the averaged values of parameters, and $\widetilde{M}_{ij}$ is the perturbation matrix. The forms of $\widehat{M}_{ij}$ and $\widetilde{M}_{ij}$ can be easily deduced using (3) and (5). In equations (5) and further, summation is assumed over repeating indices.

For $\widetilde{M}_{ij} \to 0$, one can look for a solution of (5) in the form of plane wave

$$A_j(x,z) \propto e^{ik_x x + ik_z z} \; . \tag{6}$$

To simplify the writing of the following expressions we use the notation

$$k_x = p \; ; \quad k_z = q \; . \tag{7}$$

Substituting (6) into (5) we obtain a set of algrbraic equations

$$M_{ij} A_j = 0 \; , \tag{8}$$

where the matrix M_{ij} is obtained from $\widehat{M}_{ij}$ by substituting iq for $\partial/\partial z$ and ip for $\partial/\partial x$. The requirement that the set of equations (8) has non-trivial solution, $\det M_{ij} = 0$, gives the dispersion relation (and polarization vector), which should be considered as a relation between p and q for a fixed frequency ω. The analysis of the dispersion relation shows that q^2 is a single-value function of p so that for a given p two values of q are possible which have the same magnitude and opposite signs. We mark these two branches by an index "λ".

For $\widetilde{M} \to 0$, a general solution of equations (5) can be written as a sum of plane waves

$$A_j(x,z)|_{\widetilde{M}\to 0} = \frac{1}{2\pi} \int dp e^{ipx} \sum_{\lambda} f_\lambda(p) a_j^\lambda(p) e^{iq_\lambda(p) z} \tag{9}$$

where $f_\lambda(p)$ is arbitrary function of p, and $a_j^\lambda(p)$ is the polarization vector:

$$a_j^\lambda(p) = \left(-i\frac{p^2+q_\lambda^2}{\omega\gamma_0} \; ; \quad 1 \; ; \quad - -i\frac{p^2+q_\lambda^2}{\omega\gamma_0} \frac{pq_\lambda}{p^2+\eta_0} \right) \; , \tag{10}$$

where $q_\lambda = q_\lambda(p)$ is assumed to be a function of p according to dispersion relation.

In the case $\widetilde{M}_{ij} \neq 0$, a solution of (5) can be written in the form

$$A_j(x,z) = \frac{1}{2\pi}\int dp e^{ipx} \sum_\lambda F_j^\lambda(p,z) e^{iq_\lambda(p)z} \tag{11}$$

with

$$\sum_\lambda F_j^\lambda(p,z) e^{iq_\lambda(p)z} = \int A_j(x,z) e^{-ipx} dx \ . \tag{12}$$

The representation of the field in the form (11), (12) is nothing but Fourier expansion over the x coordinate (with the coordinate z as a parameter). The way we have written the Fourier transform is determined by physical problem under consideration and is the matter of convenience. In particular, since for $\widetilde{M}_{ij} \to 0$ (11) should turn into (9) with $F_j^\lambda(p) = f_\lambda(p) a_j^\lambda(p)$ independent of z, we can expect that for small $\widetilde{M}_{ij}$ the functions $F_j^\lambda(p,z)$ will be, on the average, a slowly varying functions of z represented by

$$F_j^\lambda(p,z) \simeq a_j^\lambda(p) f_\lambda(p,z) \ . \tag{13}$$

We now put (11) into (5) and perform differentiation over x and z under the integral, neglecting the second order derivatives of $f_j^\lambda(p,z)$ over z. Then, taking the inverse Fourier transform and averaging over z we obtain

$$i v_\nu(p) \frac{\partial f_\nu(p,z)}{\partial z} = \frac{1}{2\pi} < \int\int dx d\kappa \sum_\lambda \mu^{\nu\lambda}(p,\kappa;\ x,z) f_\lambda(\kappa,z) e^{-i(p-\kappa)x - i[q_\nu(p) - q_\lambda(\kappa)]z} > \tag{14}$$

where

$$v_\nu(p) = 2q_\nu \left[\frac{(p^2+q^2)^2}{\omega^2\gamma_0^2} \frac{\eta_0}{p^2+\eta_0} + 1 \right] \ ; \quad \mu^{\nu\lambda}(p,\kappa;\ x,z) = \overline{a_i^\nu(p)} \widetilde{M}_{ij}(x,z) a_j^\lambda(\kappa) \ , \tag{15}$$

and $< ... >$ denotes an averaging over z. Using (14), (15) one can easily check that the following relation holds

$$\frac{d}{dz} \int dp \sum_\lambda v_\lambda(p) |f_\lambda(p,z)|^2 = 0 \ , \tag{16}$$

which represents the energy conservation law in the process of wave scattering.

To write a solution of equation (14) we need to set boundary conditions which follow from the scattering problem discussed above. Namely, at large positive z, the amplitudes of all harmonics corresponding to negative group velocities, ($\nu = 2$), thus negative $v_\nu(p)$, are equal to zero: $f_2(p, z \to \infty) = 0$; and at large negative z, the amplitudes of harmonics corresponding to positive $v_\nu(p)$ ($\nu = 1$) are determined from the expansion of the incident wave: $f_1(p, z \to -\infty) = f_{inc}(p)$. For both $\nu = 1$ and $\nu = 2$, the solution of equation (14) can be written in the form:

$$\begin{aligned} & i v_\nu(p) [f_\nu(p, z \to \infty) - f_\nu(p, z \to -\infty)] \\ = & \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \int\int dx d\kappa \sum_\lambda \mu^{\nu\lambda}(p,\kappa;\ x,z) f_\lambda(\kappa,z) e^{-i(p-\kappa)x - i[q_\nu(p) - q_\lambda(\kappa)]z} \end{aligned} \tag{17}$$

Equation (17) is an integral equation for the function $f_\nu(p,z)$. Since its kernel $\mu^{\nu\lambda}(p,\kappa;\ x,z)$ is in a sense small, the first approximation of iteration method gives a good idea of the solution. Thus, substituting an unperturbed spectrum (corresponding to the incident wave packet) into the right hand side of (17), that is $f_1(p,z) = f_{inc}(p)$ and $f_2(p,z) = 0$, and performing the integration over x and z we find

$$\begin{aligned} & iv_\nu(p)[f_\nu(p,z\to\infty) - f_\nu(p,z\to-\infty)] \\ = \ & \frac{1}{2\pi}\int d\kappa\mu^{\nu 1}[p,\kappa;\ p-\kappa,\ q_\nu(p)-q_1(\kappa)]f_{inc}(\kappa)\ , \end{aligned} \tag{18}$$

where $\mu^{\nu 1}[p,\kappa;\ p-\kappa,\ q_\nu(p)-q_1(\kappa)]$ is Fourier transformation of the function $\mu^{\nu 1}(p,\kappa;\ x,z)$ over space coordinates x and z corresponding to the wave numbers $p-\kappa$ and $q_\nu(p)-q_1(\kappa)$, respectively. From (18) we see that if the initial spectrum is narrow centered on p_0, the final spectrum out of the scattering region may be broadened. Indeed, the spectrum will be enriched by those wave numbers p and $q_\nu(p)$ for which $\mu^{\nu 1}[p,p_0;\ p-p_0,\ q_\nu(p)-q_1(p_0)] \neq 0$. We should stress that the spectrum is modified not only beyond the scattering region (by harmonics with positive group velocities), but also before the scattering region (by harmonics with negative group velocities). In a general sense the wave field described by the equations above can be considered as an eigen mode in plasma in presence of density irregularities.

The present theory describes in a consistent manner the scattering processes inside and between quasi-longitudinal and quasi-electrostatic (resonance) branches of whistler mode waves. These processes determine, or play a part, in the phenomena like amplitude modulation [3], [4] and apparent spectral broadening [5], [6] of VLF transmitter signals observed on satellites. Apart of this, the scattering of quasi-electrostatic waves into quasi-longitudinal ones can play a part in the problem of whistler mode wave exit from the magnetosphere into the lower ionosphere.

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