EXACT SOLUTIONS FOR WAVES IN COLD BOUNDED PLASMAS

Ming Y. Yu[1] and **Lennart Stenflo**[2]

[1]*Theoretical Physics I, Ruhr-University Bochum, D-44780 Bochum, Germany*
[2]*Department of Plasma Physics, Umeå University, S-90187 Umeå, Sweden*

Waves in low-temperature plasmas are of much recent interest because of their relevance to new sources of radiation and particle beams, signal transmission in optical fibers, as well as devices for plasma-assisted material processing. Low-temperature plasmas are usually confined by rigid dielectric or metal containers instead of strong magnetic fields as in the fusion machines. In such plasmas, boundary dependent surface, or global, modes can appear. Since very large amplitude oscillations are often excited during the production and control of the plasma, it is important to understand the nonlinear modes and their interactions in low-temperature plasmas.

It is well known [1] that exact solutions for finite amplitude waves in plasmas bounded by dielectrics can be obtained. These solutions satisfy the cold fluid equations, the Maxwell's equations, and the appropriate boundary conditions. No perturbation or truncation, or any other approximation, is used to obtain the solutions. The latter describe oscillating spatial patterns, in particular *oscillons*, which are of much recent interest not only because they represent ubiquitous physical phenomena in different branches of science [2], but are also useful for verifying new approximation or numerical schemes in the study of nonlinear effects.

Here we investigate nonlinear global plasma modes in a magnetized plasma bounded by a spherical dielectric. Our formulation can also be adapted to consider waves in any one-component non-neutral plasma [3]. Exact solutions for the nonlinear modes are obtained, and the similarity between the present investigation to earlier works is discussed.

Consider electrostatic oscillating spatial patterns in a spherical cold electron plasma in a positive background of immobile ions. The plasma does not have to be neutral and it is bounded at $r = R$ by a rigid dielectric of constant permittivity ϵ_d. An external magnetic field $B_0\hat{\mathbf{z}}$ is present. The evolution of the electron density n and the fluid velocity $\mathbf{v}$ is governed by the cold fluid equations

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0, \tag{1}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{q}{m}\mathbf{E} + \Omega \mathbf{v} \times \hat{\mathbf{z}}, \tag{2}$$

where $q = -e$ and m are the electron charge and mass, $\Omega = qB_0/m$ is the electron cyclotron frequency, and $\mathbf{E} = -\nabla\varphi$ is the wave electric field. The plasma is assumed to be of low temperature such that the pressure effects can be ignored. The potential φ satisfies the Poisson equation

$$\nabla^2\varphi = -(q/\epsilon_0)(n - n_0), \tag{3}$$

where the background ion density n_0 inside the sphere is assumed to be constant.

The approach here is similar to that of the method of separation of variables in the theory of linear partial differential equations and that for investigating nonlinear quadrupolar motion [4]. It is also similar to that used by Lorenz, who investigated nonlinear atmospheric waves and deterministic chaos by separating the spatial variations from the temporal one. However, here no *ad hoc* truncation of the higher spatial harmonics will be made.

The spatial wave structure inside ($r < R$) the plasma is assumed to be of the form

$$n = n(t), \tag{4}$$

$$\mathbf{v} = \left[(xv_m + yv_b)\hat{\mathbf{x}} + (yv_m - xv_b)\hat{\mathbf{y}} + zv_c\hat{\mathbf{z}}\right]/R, \tag{5}$$

and

$$\varphi_{r<R} = \left[(x^2+y^2-2z^2)\varphi_m + (x^2+y^2+z^2-R^2)\varphi_c\right]/R^2, \tag{6}$$

where Cartesian coordinates (x, y, z) have been used. Here, v_m, v_b, v_c, φ_m, and φ_c are functions of time only. We note that v_m and v_b correspond to the solenoidal and rotational components of the fluid velocity, respectively.

The Ansätze (4) – (6) was found by trial and error, such that the temporal and spatial dependences of the resulting equations are separable. Note that the above *Ansätze* may not be unique, and more complex structures involving higher spatial harmonics can therefore exist.

The potential inside ($r > R$) the dielectric can be of the form

$$\varphi_{r>R} = \left(1 - 3\cos^2\theta\right) R^3\varphi_m/r^3, \tag{7}$$

which satisfies the Laplace equation. That is, the corresponding electric field in the dielectric decays like r^{-3} away from the plasma boundary. Here, $r^2 = x^2 + y^2 + z^2$ and $\cos\theta = z/(x^2 + y^2 + z^2)^{1/2}$.

The nonlinear boundary condition describing the continuity of the current density across the plasma-dielectric interface is

$$\left[qnv_r - \epsilon_0\partial_t\partial_r\varphi\right]_{r=R_{-0}} = \left[-\epsilon_0\epsilon_d\partial_t\partial_r\varphi\right]_{r=R_{+0}}, \tag{8}$$

where $R_{\pm 0}$ represent $r > R$ and $r < R$ locations infinitesimally close to $r = R$. In the following we shall consider the case $\epsilon_d = 1$ (i.e., vacuum at $r > R$) without loss of generality.

The thickness (of the order of a few Debye lengths) of the boundary layer at the interface is assumed to be smaller than any other characteristic dimension of the problem, especially the attenuation length of the surface wave fields, and it is thus negligible. This thin layer also acts as a source or sink for the plasma particles during the oscillations.

Because of the Ansätze (4) – (7), the spatial and time dependences of the physical quantities are separable. Thus, substituting them into (1), (2) and (3), using the boundary condition (8), and equating terms of the same spatial dependence, one obtains

$$\dot{N} = -N(2W + V), \tag{9}$$

$$\dot{U} = -2UW + \rho W, \tag{10}$$

$$\dot{V} = -V^2 + 4\Phi_m + (N-1)/3, \tag{11}$$

$$\dot{W} = -W^2 - 2\Phi_m + U(U-\rho) + (N-1)/3, \tag{12}$$

$$\dot{\Phi}_m = N(W-V)/15, \tag{13}$$

where $N = n/n_0$, $U = v_b/\omega_p R$, $V = v_c/\omega_p R$, $W = v_m/\omega_p R$, $\Phi_m = q\varphi_m/m\omega_p^2 R^2$, and $\rho = \Omega/\omega_p$. The variable Φ_c ($= q\varphi_c/m\omega_p^2 R^2$) $= (1-N)/6$ has been eliminated. The dot represents derivative with respect to time, which is normalized by the inverse plasma frequency $\omega_p^{-1} = (m\epsilon_0/n_0 q^2)^{1/2}$. For convenience we shall normalize ω_p to unity.

The equations (9) – (13) form a set of nonlinear ordinary differential equations describing the time evolution of oscillons. They contain no arbitrary parameters other than ρ, which is a constant determined by the external magnetic field. If the vacuum at $r > R$ is replaced by a dielectric, then the dielectric constant ϵ_d also enters as an external parameter. In this case we have to replace the factor 15 in (13) by $3(2+3\epsilon_d)$.

The stationary states of (9) – (13) are given by $N = 1-2U(U-\rho)$, $W = 0$, $V = 0$, $\Phi_m = U(U-\rho)/6$, where U is a finite constant. In general, a stationary point corresponds to a steadily rotating non-neutral plasma. The equilibrium is maintained by the electric, centrifugal, as well as $\mathbf{v}\times\mathbf{B}$ forces. The corresponding plasma potential is $\varphi = \varphi_p \equiv \varphi_{m0}(3x^2+3y^2-2r^2)/R^2$, where φ_{m0} is the dimensional form of Φ_m given above. The potential at the plasma boundary is then given by $\varphi = \varphi_s \equiv \varphi_{m0}(1-3\cos^2\theta)$, which is consistent with (7).

By linearizing the evolution equations (9) – (13) about a stationary state, we can obtain the natural frequencies of the linear global plasma modes satisfying the boundary electric potential $\varphi(r=R) = \varphi_s$. Accordingly, it is easily shown that the frequencies of the linear oscillations are given by

$$\omega^2 = \frac{7}{10} + \frac{3}{5}U_0(U_0-\rho) + \frac{1}{2}\rho^2 \pm \left[\frac{9}{100} + \frac{1}{25}U_0(U_0-\rho) + \frac{\rho^2}{10} + \frac{89}{25}U_0^4 - \frac{178}{25}U_0^3\rho + \frac{134}{25}U_0^2\rho^2 - \frac{9}{5}U_0\rho^3 + \frac{1}{4}\rho^4\right]^{1/2}, \tag{14}$$

where we have eliminated N_0 in favor of U_0. Equation (14) describes two volume and two surface modes of a non-neutral plasma.

For the particular stationary state $N = 1$, $W = V = U = \Phi_m = 0$, corresponding to a neutral plasma, we obtain from (14) the characteristic frequencies

$$\omega^2 = \frac{7}{10} + \frac{\rho^2}{2} \pm \left(\frac{9}{100} + \frac{\rho^2}{10} + \frac{\rho^4}{4}\right)^{1/2}, \tag{15}$$

which describe the linear plasma and upper-hybrid volume and surface waves in the spherical geometry. In the limit $\rho \to 0$ we recover the frequencies $\omega = 1$ and $\omega = \sqrt{2/5}$ for the linear volume and fundamental surface plasma modes in an unmagnetized spherical plasma, respectively.

In general, eqs. (9) – (13) can be integrated numerically. Depending on the initial conditions and the parameter ρ, coherent and chaotic oscillations, as well as highly peaked solitary pulses, can exist. One finds solutions of many types, all of which indicate that a nonlinear interaction exists. Even for small amplitudes, weak nonlinear interaction leads to a small frequency broadening. On the other hand, there exist also solutions with large amplitudes, where the oscillations remain fairly coherent. The nonlinear oscillations are strongly coupled, but there is no resonance because the fixed spatial wave pattern. We also find highly nonlinear isolated oscillon solutions. The oscillon solutions involve variables with very large amplitudes. At present there is still very little understanding of the oscillons. For other initial parameters, highly chaotic solutions can also be found.

The equations (9) – (13) are self consistent and exact within the cold fluid model and the boundary condition of the continuity of the current density across the plasma-dielectric interface. It is of interest to point out that the plasma surface potential φ_s is similar to that of the well-known Penning trap [5] configuration and the classical ellipsoidal figures of equilibrium [4]. We note however that unlike the classical Penning trap theories, here the shape of the plasma boundary is not free but identical to the containing wall, and there is no pinching of the plasma. Equation (14) then gives the frequencies of the global electrostatic modes associated with the self-consistent trap, and (9) – (13) describe nonlinear bulk and surface waves, as well as oscillons in the plasma.

The present theory can easily be extended to investigating oscillons and global waves in non-neutral one-component plasmas. For example, by setting $n_0 = 0$ and replacing the electrons by ions, one can easily adapt (9) – (13) to a non-neutral pure ion plasma. In this case the evolution equations are modified in that the factor $N - 1$ is replaced by N, which is now the normalized ion density.

Acknowledgements. This work was partially supported by the Sonderforschungsbereich 191 Niedertemperatur Plasmen.

References

[1] S.V. Vladimirov, M.Y. Yu and V.N. Tsytovich: Phys. Reports **241**, 1 (1994); L. Stenflo: Phys. Scripta T **63**, 59 (1996), *and the references therein*.

[2] L. Stenflo and M.Y. Yu: Phys. Fluids B **1**, 1543 (1989); Nature **384**, 224 (1996), *and the references therein*.

[3] R.C. Davidson: Theory of non-neutral plasmas. Benjamin, Reading MA, 1983.

[4] S. Chandrasekhar: Ellipsoidal figures of equilibrium. Yale University Press, New Haven, 1969.

[5] D.H.E. Dubin: Phys. Fluids B **5**, 295 (1993), *and the references therein*.