THERMODYNAMIC FUNCTIONS OF A DENSE NONIDEAL PLASMA

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1. Introduction

The study of nonideal plasma phenomena is a subject of great interest for several fields of modern physics. In particular, it is important in the context of ICF research and astrophysics.

At sufficiently high electron densities for which $\Gamma > 1$ (The interaction coupling parameter Γ is defined as the ratio of the average Coulomb potential energy between particles to the average kinetic energy), classical statistical theories fail because of thermodynamic instabilities, which are inhibited by quantum effects. The classical plasma (pressure) would collapse for $\Gamma > 1$ due to the negative electron-ion interaction energy, whereas in reality the pressure remains positive in a plasma due to Fermi pressure (exclusion principle) of the electrons. For these reasons, we present here a quantum-statistical theory for nonideal plasmas based on concepts similar to those used by Debye for solids [1].

2. Description of the model

The plasma under consideration is a quasi-homogenous high pressure and fully ionized plasma consisting of electrons of charge -e and density n = N/V, and ions of charge +Ze and density n/Z.

In this Coulomb system, the electrons and ions interact through their longitudinal Coulomb field [2]. There exist N (high frequency branch) and N/Z (low frequency branch) characteristic frequencies ω_i of longitudinal oscillations. Each plasma oscillator of frequency ω_i can have the energies $E_n^i = \left(1 + \frac{1}{2}\right)\hbar\omega_i$, $n = 0,1,2,...$, so the energy of a plasma state with n plasmons of frequency ω_i is $E\{i\} = \sum_n n\hbar\omega_i$ where $\{i\}$ refers to the entire set of given eigenfrequencies ω_i [3]. Accordingly, the partition function Q of the longitudinal plasma oscillations is:

$$Q = \underset{i}{\Pi} \underset{n}{\sum} \exp\{-n\hbar\omega_i / k_B T\} = \underset{i}{\Pi}\left\{1 - \exp\left(-\hbar\omega_i / k_B T\right)\right\}^{-1} \tag{1}$$

From Q, the thermodynamic functions such as pressure, internal energy, entropy, etc... are derived in the usual way.

2.1 Dispersion relations

The high frequency branch of the space charge waves is due to longitudinal electron oscillations governed by the following dispersion relation for degenerate electrons:

$$\omega^2 = \omega_p^2\left(1 + b^2 q^2\right), \tag{2}$$

where

$$b^2 = (1 + \tfrac{5}{12}\pi^2\theta^2).\ \tfrac{3}{5}v_F^2 \big/ \omega_p^2, \tag{3}$$

v_F is the Fermi speed and ω_p is the plasma frequency.

In (3), we set $\theta = 0$ when the electrons are assumed to be highly degenerate ($\theta \le 0.1$). The low frequency branch of the space charge waves is essentially due to ion sound waves:

$$\omega = \delta(q)C_s q, \tag{4}$$

where $\delta(q)$ is a correction factor of order 1 and C_s is the speed of sound of the ions.

3. Statistical thermodynamics

3.1 The free energy (Helmoltz energy)

The resulting Hamilton function with Coulomb interaction gives for the free energy of the plasma the ideal (F_0) and (ΔF) contributions:

$$F_0 = \sum_{s=e,i} F_s^{(0)}\ ,\ \ \Delta F = \sum_{s=e,i}\widetilde{F}_s + E_M. \tag{5}$$

Here, $F_s^{(0)}$ is the ideal free energy of the interacting plasma components s.

In this approach, the free interaction energy is due to the static Coulomb interaction of the electrons and ions in their 'equilibrium positions' E_M (Madelung energy) and their oscillation energies about average equilibrium positions $\widetilde{F}_s$ (plasmon energies).

The free energy of the plasmons is:

$$\widetilde{F}_s / k_B T (V/2\pi^2) = \int_0^{\hat{q}_s} \mathrm{Log}\big[1 - \exp\{-\hbar\omega(q)/k_B T\}\big] q^2 dq. \tag{6}$$

whereas the high frequency contribution $\widetilde{F}_e$ is given by:

$$\widetilde{F}_e = N k_B T\left[\mathrm{Log}\left\{1 - \exp - \varepsilon_p\left(1 + b^2\hat{q}_e^2\right)^{1/2}\right\} - \mathcal{F}\left(\varepsilon_p, b\hat{q}_e\right)\right], \tag{7}$$

where

$$\mathcal{F}\left(\varepsilon_p, b\hat{q}_e\right) = \left(b\hat{q}_e\varepsilon_p\right)^{-3}\int_{\varepsilon_p}^{\hat{\varepsilon}_e}\frac{\left(\varepsilon^2 - \varepsilon_p^2\right)^{3/2}}{\exp(\varepsilon) - 1}d\varepsilon, \tag{8}$$

$\varepsilon_p = \hbar\omega_p / k_B T$, $\hat{\varepsilon}_e = \varepsilon_p\left(1 + b^2\hat{q}_e^2\right)^{1/2}$ and $\hat{q}_e = \left(6\pi^2 n\right)^{1/3}$ is the electron wave-number limit.

The low frequency contribution $\widetilde{F}_i$ is given by:

$$\widetilde{F}_i = \frac{N}{Z} k_B T \left[\mathrm{Log}\left\{ 1 - \exp\left(-\hbar C_s / k_B T \right) \delta\left(\hat{q}_i \right) - G\left(\hat{q}_i \right) \right\} \right] \tag{9}$$

$$G(\hat{q}_i) = G(\hat{\varepsilon}_i) = \hat{\varepsilon}_i^{-3} \int_0^{\hat{\varepsilon}_i} \frac{\varepsilon^3}{e^\varepsilon - 1} d\varepsilon \tag{10}$$

$$= \frac{1}{3}\left[1 - \frac{3}{8}\hat{\varepsilon}_i + \frac{1}{20}\hat{\varepsilon}_i^2 + ... \right] \quad \text{when} \quad \hat{\varepsilon}_i \ll 1$$
$$= \frac{1}{15}\pi^4 \hat{\varepsilon}_i^{-3} + O\left\{ \exp(-\hat{\varepsilon}_i) \right\} \quad \text{when} \quad \hat{\varepsilon}_i \gg 1 \tag{11}$$

$\hat{\varepsilon}_i = \hbar\omega_i / k_B T = \hbar C_s \hat{q}_i / k_B T$ and $\hat{q}_i = \left(6\pi^2 n / Z \right)^{1/3}$ is the ion wave- number limit.

Quasi-lattice energy E_M

The equilibrium positions of the electrons and ions about which the electrostatic oscillations occur, form an electron lattice and ion lattice with an incomplete ordering. In the Wigner-Seitz approximation, (ion sphere model), we have:

$$E_M = -\left(\frac{3}{4\pi} \right)^{1/3} \frac{N k_B T}{Z^{5/3}} 0.9\Gamma \tag{12}$$

3.2 The equation of state (Pressure)

In the same way, we deduce the pressure: $\quad \Delta P = -\left. \frac{\partial(\Delta F)}{\partial V} \right|_T \tag{13}$

Where the high frequency contribution is given by :

$$\widetilde{P}_e / n k_B T = 2\left(\varepsilon_p b \hat{q}_e \right)^{-3} \left[\left\{ 1 - \frac{5}{12}\frac{\pi^2\theta^2}{1 + \frac{5}{12}\pi^2\theta^2} \right\} I_{3/2} + \frac{3}{4}\varepsilon_p^2 I_{1/2} \right] \;, \tag{14}$$

where $\theta = 0$ is set for electrons assumed to be highly degenerate ($\theta \leq 0.1$).

$$I_n = \int_{\varepsilon_p}^{\hat{\varepsilon}_f} \frac{\left(\varepsilon^2 - \varepsilon_p^2 \right)^n}{\exp(\varepsilon) - 1} d\varepsilon, \qquad n = 3/2, \; 1/2 \tag{15}$$

And the low frequency contribution: $\widetilde{P}_i / n k_B T = \frac{1}{Z}G(\hat{\varepsilon}_i) \tag{16}$

The Madelung contribution: $\quad P_M / n k_B T = \frac{1}{3}\left(-0.9\Gamma / Z \right)$ for $\Gamma > 1 \tag{17}$

3.3 The internal energy

The internal energy is given by:
$$\Delta U = -k_B T^2 \frac{\partial}{\partial T}\left(\frac{\Delta F}{k_B T}\right), \tag{18}$$

where the high frequency contribution:
$$\widetilde{U}_e / Nk_B T = 3\left(\varepsilon_p b \hat{q}_e\right)^{-3}\left[cI_{3/2} + \varepsilon_p^2 I_{1/2}\right] \tag{19}$$

and the low frequency contribution:
$$\widetilde{U}_i / Nk_B T = \frac{3}{2Z} G(\hat{\varepsilon}_i). \tag{20}$$

For the Madelung contribution:
$$U_M / Nk_B T = \left(-0.9\Gamma / Z\right) \text{ for } \Gamma > 1 \tag{21}$$

4. Results and comparisons with existing theories

Table 1 shows a comparison of the present results with several existing models over a wide range of plasma conditions. It is shown that the present collective approach describes fairly well nonideal effects of strongly coupled plasma; the low frequency branch (ions) oscillations contributing the most to the final results of the thermodynamic functions.

Γ	this theory	MC [4]	$q_{TF.}$ exp. [6]	Pert. [6]	Var. OCP [6]	Var. HS1 [5]	Var. HS2 [5]	[7]
2	0.305 1.358	0.442 1.358	0.437 1.363	0.438 1.370	0.436 1.360	0.338 1.139	0.370 1.442	0.435 1.355
6	1.532 4.997	1.524 4.645	1.516 4.647	1.516 4.647	1.517 4.645	1.386 4.367	1.425 4.463	
10	2.749 8.623	2.573 8.071	2.565 8.069	2.565 8.072	2.567 8.068	2.498 7.753	2.538 7.854	2.654 8.082
20	5.777 17.666	5.540 16.802	5.530 16.791	5.530 16.797	5.532 16.794	5.353 16.408	5.384 16.512	
100	29.871 89.806	29.127 88.068	29.113 87.983	29.109 88.051	29.110 87.972	28.836 87.364	28.873 87.459	

Table 1. Comparison of results of various theories for Z=1 and r_s = 0.1 ($r_s = a/a_0 Z^{1/3}$, a is the interionic distance and a_0 is the Bohr radius). For each value of Γ, the first line lists the results of negative pressure and the second line the negative energy.

References

[1] P. Debye and E. Hueckel: Z. Phys. **24**, 185 (1923).

[2] A.H. Khalfaoui: IEEE Trans. on Plasma Science, vol. PS-12, n°3 (1984).

[3] H.E. Wilhelm and A.H. Khalfaoui: Aust. J. Phys. **35**, 425 (1982).

[4] DeWitt and Hubbard: Astrophys. J. **205,** 295 (1976).

[5] M. Ross and D. Seale: Phys. Rev. A **9**, 396 (1974).

[6] S. Galam and J.P. Hansen: Phys. Rev. A **14**, 824 (1976).

[7] J. P. Hansen and R. Mc Donald: Phys. Rev. A **23**, 2046 (1981).