ON THE RESONANT INTERACTIONS BETWEEN NONLINEAR WAVES

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Plasma is a highly nonlinear medium that can support a wide variety of nonlinear wave phenomena. An important one among these, with many applications, is the resonant interactions between three and four waves. The three wave mixing was intensively studied for electronic and Alfvén waves [5,6,8], while the four wave mixing was treated for the first time in plasma by Rauf [4], for nonlinear dissipative Alfvén waves. In [1], the problem of resonant interactions between two different types of modes (the Alfvén and acoustic ones) is discussed.

In the present paper we examine the resonant interactions between four nonlinear Alfvén waves propagating through ideal, relativistic electron-positron plasma. After that we study the interaction between magnetoacoustic waves in ideal MHD plasma, when the fundamental component resonantly interacts with its higher harmonic components. In the end the two situations are compared.

i) In order to analyse the first situation, we start from the nonlinear evolution equation for Alfvén waves propagating through relativistic electron-positron plasma [7]:

$$\frac{\partial \Phi}{\partial t} + \frac{\partial (|\Phi|^2 \Phi)}{\partial z} + \frac{\partial^3 \Phi}{\partial z^3} = 0, \qquad (1)$$

where $\Phi = b_x + i b_y$ denotes a linear combination of magnetic field components transverse to the propagating axis Oz. All the variables in (1) are in normalised form (The equation (1) is derived from the two fluid plasma model by rescaling the independent variables and using the ordering expansion for the dependent ones [7]).

The equation (1) admits a harmonic solution:

$$b_x = a\cos(kz - \omega t),\ \ b_y = a\sin(kz - \omega t) \qquad (2)$$

($a -$arbitrary constant), provided ω obeys a nonlinear dispersion law:

$$\omega = k(a^2 - k^2)\ . \qquad (3)$$

To study four wave mixing, we assume ϕ to be a sum of four time harmonic solutions:

$$\Phi = \sum_{j=1}^{4} a_j \phi_j (z,t) \exp[i(k_j z - \omega_j t)] \qquad (4)$$

where (a_j, ω_j, k_j) are connected through (3) and ϕ_j (j=$\overline{1,4}$) are complex functions. The frequencies ω_j and the wavenumbers k_j are chosen to satisfy the resonant conditions:

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \qquad k_1 + k_2 = k_3 + k_4. \tag{5}$$

There are several ways in which (3) and (5) could simultaneously be satisfied. For example:

a) $\omega_j = k_j$; $a_j = \sqrt{1 + k_j^2}$ (case corresponding to left circularly polarised Alfvén waves),

b) $\omega_j = -k_j$; $a_j = \sqrt{k_j^2 - 1}$ (corresponding to right circularly polarised Alfvén waves).

We shall consider only the case a) as the other one leads to similar results.

The four fundamental waves with the frequencies $\omega_1, \omega_2, \omega_3, \omega_4$ will mix together generating secondary waves, having all sum and difference frequencies. We are interested in the evolution of the fundamental waves. Inserting (4) into (1) and making use of (3) and (5), in the stationary case ($\frac{\partial}{\partial t} = 0$), we obtain:

$$\frac{\partial^3 \phi_1}{\partial z^3} - 3k_1^2 \frac{\partial \phi_1}{\partial z} + 3ik_1 \frac{\partial^2 \phi_1}{\partial z^2} + \frac{2}{a_1}\frac{\partial}{\partial z}\left(\phi_2^* \phi_3 \phi_4\right) + \frac{2ik_1}{a_1} a_2^* a_3 a_4 \phi_2^* \phi_3 \phi_4 - ik_1 |a_1|^2 \phi_1 = 0, \tag{6}$$

$$\frac{\partial^3 \phi_2}{\partial z^3} - 3k_2^2 \frac{\partial \phi_2}{\partial z} + 3ik_2 \frac{\partial^2 \phi_2}{\partial z^2} + \frac{2}{a_2}\frac{\partial}{\partial z}(\phi_1^* \phi_3 \phi_4) + \frac{2ik_2}{a_2} a_1^* a_3 a_4 \phi_1^* \phi_3 \phi_4 - ik_2 |a_2|^2 \phi_2 = 0, \tag{7}$$

$$\frac{\partial^3 \phi_3}{\partial z^3} - 3k_3^2 \frac{\partial \phi_3}{\partial z} + 3ik_3 \frac{\partial^2 \phi_3}{\partial z^2} + \frac{2}{a_3}\frac{\partial}{\partial z}(\phi_4^* \phi_1 \phi_2) + \frac{2ik_3}{a_3} a_4^* a_1 a_2 \phi_4^* \phi_1 \phi_2 - ik_3 |a_3|^2 \phi_3 = 0, \tag{8}$$

$$\frac{\partial^3 \phi_4}{\partial z^3} - 3k_4^2 \frac{\partial \phi_4}{\partial z} + 3ik_4 \frac{\partial^2 \phi_4}{\partial z^2} + \frac{2}{a_4}\frac{\partial}{\partial z}(\phi_3^* \phi_1 \phi_2) + \frac{2ik_4}{a_4} a_3^* a_1 a_2 \phi_3^* \phi_1 \phi_2 - ik_4 |a_4|^2 \phi_4 = 0, \tag{9}$$

The above nonlinear coupling equations were numerically solved to calculate the steady state spatial profile of the waves (described by $|\phi_j|$) for different initial conditions at the boundary z=0. The numerical analysis shows that, indifferent of the initial conditions, in certain plasma region, the number of waves involved in resonant interactions (emphasised by $|\phi_1|$, $|\phi_2|$, $|\phi_3|$, $|\phi_4|$) suddenly increases (see Fig.1). (Explosive spatial instabilities develop.)

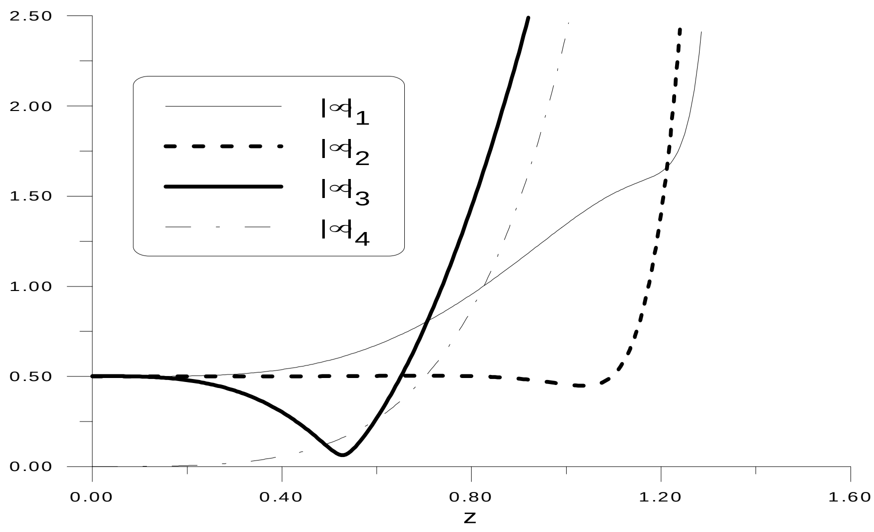

Figure 1. The steady-state spatial profile of nonlinear Alfvén waves.

This result is of importance in understanding certain aspects of pulsars. Thus, the nonlinear Alfvén waves, due to resonant interaction leads to a great number of particles with high energy, as the emission processes of pulsars showed it [9].

ii) Now we shall examine the resonant interactions between magnetoacoustic waves. We start from the fundamental MHD equations written in dimensionless form [1] and use the derivative expansion method [3]. The medium perturbations are expanded into asymptotic series [3]:

$$\tilde{f} = f_0 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + ... \quad (\varepsilon \text{ -dimensionless parameter }, \varepsilon < 1). \qquad (10)$$

In order to describe the resonant interactions between the fundamental magnetoacoustic mode and its higher harmonic components, in the first order approximation (in ε) we seek solution of the type:

$$b_x^{(1)} = b_1 e^{i\theta} + b_2 e^{2i\theta} + b_3 e^{3i\theta} + b_4 e^{4i\theta} + c.c., \qquad (11)$$

with $b_j \left(j = \overline{1,4}\right)$ – linear complex amplitude and $\theta = kz - \omega t$.

If in the linear approximation, the four harmonic components with ω, 2ω, 3ω, 4ω are independent each other, in the second order approximation, a coupling between the four harmonics develops. Thus, working with the set of equations in ε^2, imposing the nonsecularity condition on $\exp(\pm i\theta)$, $\exp(\pm 2i\theta)$, $\exp(\pm 3i\theta)$, $\exp(\pm 4i\theta)$ terms in the equation which describes $b_x^{(2)}$ variable, we find the following interaction equations in a frame moving with the phase velocity, v_f:

$$i\frac{\partial b_1}{\partial t} - \left(1 + \frac{c_A^2}{v_f^2}\right)\omega(b_2 b_1^* + b_3 b_2^* + b_4 b_3^*) = 0, \qquad (12)$$

$$i\frac{\partial b_2}{\partial t} - 2\left(1 + \frac{c_A^2}{v_f^2}\right)\omega(b_1{}^2 + 2b_3 b_1^* + b_4 b_2^*) = 0, \qquad (13)$$

$$i\frac{\partial b_3}{\partial t} - 2\left(1 + \frac{c_A^2}{v_f^2}\right)\omega(4b_1 b_2 + 3b_4 b_1^*) = 0, \qquad (14)$$

$$i\frac{\partial b_4}{\partial t} - 2\left(1 + \frac{c_A^2}{v_f^2}\right)\omega(4b_1 b_3 + b_2{}^2) = 0, \quad (c_A - \text{the Alfvén velocity}). \qquad (15)$$

These equations were numerically solved and after that the time dependencies of the magnetic energy densities $\left|b_1(\omega)\right|^2$, $\left|b_2(2\omega)\right|^2$, $\left|b_3(3\omega)\right|^2$, $\left|b_4(4\omega)\right|^2$ were plotted (see fig.2). As one can observe from Fig.2, like in the precedent case, the resonant interactions between magnetoacoustic lead to "explosive" instabilities.

This result is of interest in Earth' ionosphere and magnetosphere regions which contain magnetoacoustic waves with large amplitude [2]. One of the mechanisms responsible of their appearance may be the resonant interaction process described above.

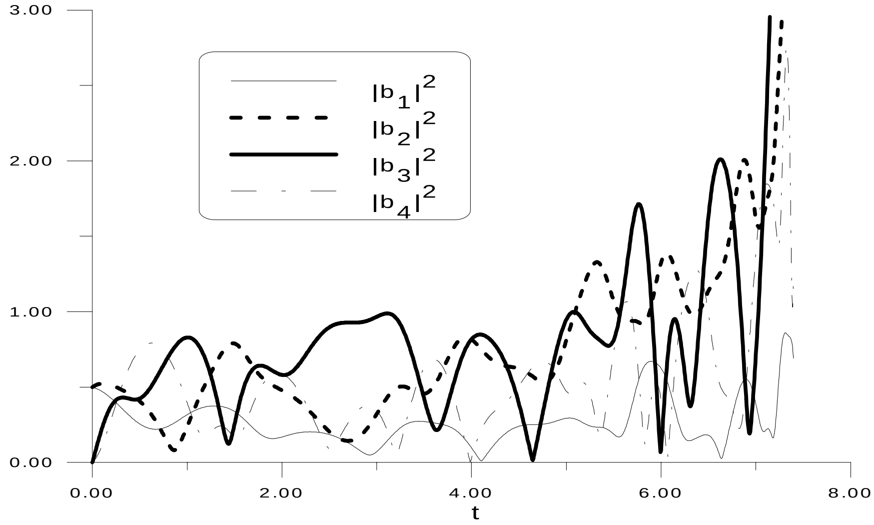

Figure 2. The time dependencies of the magnetic energy densities.

In conclusion, the curves given by Fig.1 have similar forms with that given Fig.2. Therefore, both studied situations reflect a development of "explosive" instabilities, results that can explain some astrophysical observations.

References

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