RELAXATION OF A NONIDEAL PLASMA WITH MASS FLOW IN A GRAVITATING SYSTEM

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Introduction. A convenient approximation to the space magnetohydrodynamics (MHD) is the approximation of reduced MHD [1]. Earlier it has been employed analytically and numerically for description of an equilibrium and dynamical evolution of a plasma medium [2,3], including a gravitational field case. In this paper we present elementary arguments, according to which well-known system of boundary conditions and account of a dissipation we conclude that all solutions on long times aim to the state with a zero magnetic field and zero flow. This quiescent state can be identified with a state of a hydrostatic equilibrium. Our demonstration is limited to the case of uniform, scalar conductivity and viscosity. We investigate the evolution of a cylindrical plasma column, which is ideally homogeneous and is infinite along an axis *z*. It is known that such idealization gives the elementary self-consistent model for space gravitating systems assisting to understand, in basic, features of it’s behavior.

The reduction of the MHD equations. We shall consider a dynamics of plasma structures, in which, because of a large size, the dispersion is negligibly small, so they can be described within the framework of the equations of one-fluid non-ideal MHD:

$$\partial \vec{v}/\partial t + \left[curl\,\vec{v}, \vec{v}\right] = -\nabla\left(p/\rho + \vec{v}^2/2 + U\right) + \left[curl\,\vec{B}, \vec{B}\right]/(4\pi\rho) + \nu\,\Delta\vec{v}\,, \quad (1)$$

$$\partial\vec{B}/\partial t = curl\left[\vec{v}, \vec{B}\right] + \mu\,\Delta\vec{B}, \quad (2)$$

$$div\;\vec{v} = 0, \quad (3)$$

$$div\;\vec{B} = 0, \quad (4)$$

$$\Delta U = -4\pi\,G\rho, \quad (5)$$

written for an incompressible plasma ($\rho = const$). In this case the pressure may be considered as a gauge function, such as the Lagranges factor, appropriating to restriction (3) and it can be eliminated from Eq. (1) by the differential operator *curl*.

Here the following notations are used: ρ - total mass density, $\vec{v}$ - hydrodynamic velocity, $\vec{B}$ - magnetic field, p - pressure of plasma, U - gravitational potential, G - gravitational constant, ν and μ - kinematic and magnetic viscosity.

Following [4], we shall consider translationally symmetrical solutions ($\partial/\partial z = 0$) of the system Eqs. (l-5) in a cylindrical coordinates (r, θ, z), assuming, without loss of generality, that longitudinal component (along *z* - axis) of a velocity and magnetic field are equal to zero. For description of two-dimensional motions of plasma we shall enter Stokes’s potentials φ and ψ

$$\vec{v} = \left[\nabla\varphi\,, \vec{e}_z\right],\; \vec{B} = \left[\nabla\psi\,, \vec{e}_z\right], \quad (6)$$

where $\varphi = \varphi\,(t, r, \theta\,), \psi = \psi\,(t, r, \theta\,)$ -scalar functions of time and polar coordinates in a picture plane, and $\vec{e}_z$ is the unit vector in the *z* - direction.

Taking into account the representations (6) after some manipulation the Eqs. (l)-(4) can be reduced to the following scalar partial differential equations

$$\frac{\partial}{\partial t}\omega + \{\omega, \varphi\} = \frac{1}{4\pi\rho}\{j, \psi\} + \nu\,\Delta_\perp \omega, \tag{7}$$

$$\frac{\partial \psi}{\partial t} = \{\varphi, \psi\} - \mu\, j, \tag{8}$$

$$\frac{\partial}{\partial r}\left(\frac{p}{\rho} + \frac{\vec{v}^2}{2} + U\right) + \frac{1}{4\pi\rho}\frac{\partial \psi}{\partial r}\Delta_\perp \psi - \frac{\partial \varphi}{\partial r}\Delta_\perp \varphi = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial \varphi}{\partial t} - \nu\,\Delta_\perp \varphi\right), \tag{9}$$

$$\left\{\frac{p}{\rho} + \frac{\vec{v}^2}{2} + U, \psi\right\} = \left(\frac{\partial \psi}{\partial t} - \mu\,\Delta_\perp \psi\right)\Delta_\perp \varphi - \left(\nabla_\perp\left(\frac{\partial \varphi}{\partial t} - \nu\,\Delta_\perp \varphi\right), \nabla_\perp \psi\right), \tag{10}$$

where $j = -\Delta_\perp \psi$ - current density, $\omega = -\Delta_\perp \varphi$ - vorticity of flow,

$$\Delta_\perp = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \quad \{\varphi, \psi\} = \frac{1}{r}\left(\frac{\partial \varphi}{\partial r}\frac{\partial \psi}{\partial \theta} - \frac{\partial \varphi}{\partial \theta}\frac{\partial \psi}{\partial r}\right).$$

Boundary conditions: $\varphi = \psi = 0$ at $r = a$, which are equivalent to equalities $\mathrm{v}_r = 0$, $\mathrm{B}_r = 0$ at $r = a$.

Partial analitical solutions. Let's consider some cases of simplification for a set of equations (7) - (10).

1°. $\partial/\partial t = \mu = \nu = \vec{v} = 0$. In this case the system of Eqs. (7) - (10) collapses to the Grad-Shafranov equation

$$\Delta_\perp \psi = -\frac{1}{4\pi}\frac{d}{d\psi}(p + \rho U) \tag{11}$$

and the Strauss equation

$$\{\psi, j\} = 0. \tag{12}$$

2°. $\partial/\partial t = \mu = \nu = 0$, $\vec{v} \neq 0$. The Eqs. (7) - (10) are reduced to the following system of nonlinear equations of the form:

$$\{\varphi, \omega\} = \frac{1}{4\pi\rho}\{\psi, j\}, \qquad \{\varphi, \psi\} = 0, \tag{13}$$

$$\frac{d}{d\psi}\left(\frac{p}{\rho} + \frac{\vec{v}^2}{2} + U\right) + \frac{1}{4\pi\rho}\Delta_\perp \psi - \frac{d\varphi}{d\psi}\Delta_\perp \varphi = 0$$

If the potentials φ and ψ are related by

$$\varphi = \alpha\,(4\pi\rho)^{-1/2}\,\psi, \tag{14}$$

where $\alpha = \text{const}$, the system (7) - (10) has a solution in the form [5]:

$$\Delta_\perp \psi = F(\psi), \tag{15}$$

$$\frac{p}{\rho} + \frac{\vec{v}^2}{2} + U - \frac{\alpha^2 - 1}{4\pi\rho}\int F(\psi)\,d\psi = const,$$

where $F(\psi)$ - arbitrary function of argument ψ. In the case $\alpha = 0$ the solution (15) describes magnetostatic equilibrium (see. 1°). It follows from (15) that there is a steady flow for $\alpha^2 > 1$, starting from some threshold value of the pressure.

The solution (15) describes arbitrary dynamic equilibrium of plasma (both with magnetic islands, and with one extremum of a magnetic potential [4,5]), at which plasma flows along the magnetic surfaces $\psi = \text{const}$.

If

$$F(\psi) = -\beta^2 \psi, \tag{16}$$

then the solution for ψ takes the form

$$\psi = const\left[J_0(\kappa_m r/a) + b J_m(\kappa_m r/a)\cos m\theta\right], \tag{17}$$

where J_m - the Bessel function of first kind of order m, b- constant, $\beta = \kappa_m/a$, κ_m - the first solution of an equation $J_m(x) = 0$. Other equilibrium values are determined from (5), (13) and (14).

In the case $\alpha = \pm 1$ the Chandrasekhar's solution follows from (14),(15)

$$\vec{v} = \pm \vec{B}\,(4\pi\rho)^{-1/2}, \tag{18}$$

$$p/\rho + \vec{v}^2/2 + U = const, \tag{19}$$

independent of the form of the potential ψ.

3°. $\mu = \nu = 0$, $\partial/\partial t = -\Omega\,\partial/\partial\theta$. Let the plasma flow has a steady state of rotation. We search for a stationary solution of the equations (7) - (10) in a form of nonlinear wave, rotating with a constant angular velocity $\varphi = \varphi(r, \theta - \Omega t)$, $\psi = \psi(r, \theta - \Omega t)$, where Ω - angular velocity. Then equations (7), (8) become

$$\{\Phi, \Delta_\perp \Phi\} = \{A, \Delta_\perp A\}, \tag{20}$$

$$\{\Phi, A\} = 0, \tag{21}$$

where

$$A = \psi(4\pi\rho)^{-1/2}, \qquad \Phi = \varphi + \Omega r^2/2. \tag{22}$$

The equations (20), (21) admit the solutions in the form of localized vortices. It is possible to obtain a partial solution for (20), (21), choosing $\Delta_\perp \Phi$ and A in the form [6]

$$\Delta_\perp \Phi = -b^2\Phi + f(r), \tag{23}$$

$$A = c\Phi,$$

where b and c - constants, f - arbitrary function of r.

Taking into account boundary conditions the solution (20), (21) can be represented in the form:

$$\psi = (4\pi\rho)^{1/2}\left[J_0(\kappa_m r/a) + b J_m(\kappa_m r/a)\cos m(\theta - \Omega t)\right], \tag{24}$$

$$\varphi = \alpha\left[J_0(\kappa_m r/a) + b J_m(\kappa_m r/a)\cos m(\theta - \Omega t)\right] + \Omega\left(r^2 - a^2\right)/2.$$

4°. Let's consider a special case of a diffusion-like solutions, when the potentials of a velocity and magnetic field are restricted with

$$\frac{\partial \varphi}{\partial t} - \nu \Delta_\perp \varphi = 0, \quad \frac{\partial \psi}{\partial t} - \mu \Delta_\perp \psi = 0. \tag{25}$$

In this case system (7) - (10) is transformed to

$$\{\varphi,\psi\}=0, \qquad \{p/\rho+\vec{v}^2/2+U,\psi\}=0,$$

$$\frac{\partial\psi}{\partial r}\left[\frac{\partial\psi}{\partial t}\frac{1}{4\pi\rho\mu}+\frac{d}{d\psi}\left(\frac{p}{\rho}+\frac{\vec{v}^2}{2}+U\right)\right]=\frac{\partial\varphi}{\partial r}\frac{1}{\nu}\frac{\partial\varphi}{\partial t}. \qquad (26)$$

The solution in this case takes the form

$$\psi=(4\pi\rho)^{1/2}\left[J_0(\kappa_m r/a)+bJ_m(\kappa_m r/a)\cos m\theta\right]\exp\left(-\kappa_m^2\mu t/a^2\right), \qquad (27)$$

$$\varphi=\alpha\left[J_0(\kappa_m r/a)+bJ_m(\kappa_m r/a)\cos m\theta\right]\exp\left(-\kappa_m^2\nu t/a^2\right). \qquad (28)$$

The equations (25)-(26) describe a relaxation of non-ideal plasma and the second equation (26) gives the known condition of a hydrostatic equilibrium of gravitating systems

$$\nabla(p/\rho+U)=0 \qquad (29)$$

We can remark, that the relaxation of a velocity and magnetic field develop with different dumping rates. Since $\kappa_0<\kappa_m$ for all *m*, plasma finally relaxes to the radially symmetrical state given by

$$\psi=(4\pi\rho)^{1/2}J_0(\kappa_0 r/a)\exp\left(-\kappa_0^2\mu t/a^2\right), \qquad (30)$$

$$\varphi=\alpha J_0(\kappa_0 r/a)\exp\left(-\kappa_0^2\nu t/a^2\right).$$

Conclusions. It is shown, that the system of the reduced MHD-equations permits a realization of various stationary states: a static equilibrium, stationary flow and equilibrium of rotating plasma as a nonlinear wave. However, the stability problem of obtained dynamic equilibrium states is not yet investigated. For this reason it is impossible to make any conclusion about a possibility of realization of all these states in a nature.

In the present work we also pay attention to the problem of finding of exact solutions of the reduced MHD equations for non-ideal plasma. We have obtained a new class of diffusion-like solutions corresponding to slowly varying states. At initial time, due to a small magnitudes ν and μ, given by the inequalities $\kappa_m^2\mu t/a^2<<1$, $\kappa_m^2\nu t/a^2<<1$, diffusion-like solutions are well match with solutions for stationary flow and do not match with solutions such as a nonlinear wave.

In addition, we have shown, that on long times diffusion-like solutions tend to stationary states with both zero flow and magnetic field. This result is in agreement with the result of the work [3], in which behavior on a long time of reduced MHD was also analyzed.

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