ANALYTICAL STUDY OF MAGNETIC STRUCTURES IN 'MHD': METHODOLOGIES

L. J. Palumbo and C. Chiuderi

Dip. di Astronomia e Scienze dello Spazio, Largo E. Fermi 5, I-50125 Firenze, Italy

Abstract

We perturb a generical ideal MHD static equilibrium with traslational symmetry by employing the normal modes method. The applied perturbation has the same symmetry as the equilibrium state and it is incompressible. A scalar fourth order differential equation is obtained for the Stokes function of the velocity. Using the Green's function technique this is cast into the form of an integro-differential equation. A 'perturbation' operator can be defined, whose eigenvalues are the squares of the perturbation's oscillation frequencies (ω^2). This operator proves to be self-adjoint and its eigenvalues have no definite sign *i.e.* both stable and unstable solutions are admitted. However, a sufficient stability criterium that depends on non perturbed quantities only is stablished. We apply our perturbation operator to three different equilibrium states: two of them are force-free (with both constant and non-constant α), while the third one has a resulting force. We fully solve, analitically, the resulting eigenvalue equations by employing the integral method.

1. Perturbation of equilibria with a symmetry

We adopt as our departure equations system, the equations that describe an MHD ideal plasma [1]. We employ the cartesian coordinates and impose traslational symmetry, taking *z* as the ignorable coordinate. This choice enables us to reduce our former system of equations to four scalar equations by means of:

a) The definition of the Stokes functions for the divergence-free quantities.

b) The use of the Jacobian formalism to get a compact presentation of our problem.

So, our equations system remains:

$$\frac{\partial B_z}{\partial t} + [\psi, v_z] + [B_z, \chi] = 0 \ ,$$
$$\frac{\partial \psi}{\partial t} - [\chi, \psi] = 0 \ ,$$
$$\frac{\partial \Omega_z}{\partial t} + [\Omega_z, \chi] - \frac{1}{c}\left[\frac{J_z}{\rho}, \psi\right] + \frac{1}{4\pi}\left[\frac{B_z}{\rho}, B_z\right] - \left[P, \frac{1}{\rho}\right] = 0 \ ,$$
$$\frac{\partial v_z}{\partial t} - [\chi, v_z] - \frac{1}{4\pi\rho}[B_z, \psi] = 0 \ .$$

$$\text{where :} \ \Omega = \nabla \times \mathbf{v} \ \ \rightarrow \Omega_z = -\nabla^2 \chi \ , \qquad \chi : \ \text{Stokes function of the velocity} \ ,$$
$$J_z = -\frac{c}{4\pi}\nabla^2\psi \ , \qquad \psi : \ \text{Stokes function of the magnetic field} \ ,$$
$$[F, G] = \text{Jacobian of } F \text{ with } G \ .$$

Now we perturb and linearize our equations, setting:

$$P = P_0 + P_1 \ , \quad \rho = \rho_0 + \rho_1 \quad \rightarrow \quad \rho = \rho_0 = costant$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 \ , \quad \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 \quad \rightarrow \quad \mathbf{v}_0 = 0$$

where the index 0 indicates the equilibrium state and the index 1 denotes a small perturbation. Now, a single differential equation for χ_1 is reached. Finally, developing $\chi_1(x, y, t)$ in normal modes: $\chi_1 = \hat{\chi}\,(x, y)\, exp\,(-i\omega t)$, the normal modes equation is obtained:

$$\omega^2 \nabla^2 \hat{\chi} + \left[\frac{\psi_0}{4\pi\rho_0}, \left[\psi_0, \nabla^2 \hat{\chi} \right] + 2 \left[\frac{\partial \psi_0}{\partial x}, \frac{\partial \hat{\chi}}{\partial x} \right] + 2 \left[\frac{\partial \psi_0}{\partial y}, \frac{\partial \hat{\chi}}{\partial y} \right] \right] = 0 \ , \tag{1}$$

or, in a more compact form:

$$\omega^2 \hat{\chi} = L \hat{\chi} \ , \tag{2}$$

where L is a fourth order differential operator.

2. Properties of the perturbation operator

2.1 Proof of the hermiticity of L

We show that L is a self-adjoint operator, following the next steps:

- We generalize a method from Sedláček [2], which mainly consists in the introduction of a scalar product defined *ad-hoc*, so as L results to be hermitian in it. We try with:

  $$\langle f, g \rangle = \Big(f, \nabla^2 g \Big) = \int dxdy \ f \overline{\nabla^2 g}$$

  , where the Laplace operator, ∇^2, works as a 'weight' operator and is shown to be self-adjoint when applied to functions whose value, or whose normal derivative, is zero at the boundary(*). These boundary conditons will be adopted in the following.

- We show that $\langle , \rangle$ is symmetrical, bilinear and has non-zero norm, this means it is a scalar product.

- Taking into account property (*), we finally show that:

  $$\langle f, Lg \rangle = \langle Lf, g \rangle.$$

  So, L results to be self-adjoint and ω^2 is real.

2.2 Determination of a stability criterium

Taking the scalar product of equation 2 with $\hat{\chi}$ we get:

$$\omega^2 = \frac{\langle \hat{\chi}, L\hat{\chi} \rangle}{\langle \hat{\chi}, \hat{\chi} \rangle} = -|\omega|^2 \frac{\left[\langle F'(\psi_0) |\psi_1|^2 \rangle - \langle B_{1x}^2 + B_{1y}^2 \rangle \right]}{\langle \hat{\chi}, \hat{\chi} \rangle}$$

where $\langle \hat{\chi}, \hat{\chi} \rangle < 0$ because of the imposed boundary conditions.

This is a sufficient stability criterium which involves non-perturbed quantites only, because, in principle, it is enough to know the sign of $F'(\psi_0)$ *i.e.* J_z'.

2.3 Connection with the energy method

By the definition of a displacement vector ξ_1 such as $\mathbf{v}_1 = \frac{\partial}{\partial t} \xi_1$, we show that:

$$- \int dxdy (\xi_{1_\perp} \cdot \frac{\partial^2}{\partial t^2} \bar{\xi}_{1_\perp}) = \delta W_\perp \ ,$$

where $\delta W_\perp$ is the work done by the forces generated from the perturbation perpendicular to *z*. On the other hand, it results that:

$$-\omega^2\langle\eta,\eta\rangle = \delta W_\perp \ ,$$

with η: Stokes funtion of ξ_1 and $-\langle\eta,\eta\rangle > 0$. Thus, we finally get:

$$\begin{aligned}\delta W_\perp > 0 \Rightarrow \omega^2 > 0 \text{ which implies stability}\\ \delta W_\perp < 0 \Rightarrow \omega^2 < 0 \text{ which implies instability} \ .\end{aligned}$$

Energy method [3]: the perturbations are unstable when they make the total work $\delta W < 0$.
Our application of the normal modes method: it is enough to verify that $\delta W_\perp < 0$ to know that the equilibrium is unstable. This condition is less restrictive than the one given above.

3. Examples

Three examples of equilibria that verify the imposed conditions are:

$$\begin{aligned}&a) \qquad \psi_0(y) = \frac{B_0}{a}\cos ay \ \text{ : force free of constant } \alpha\\ &b) \qquad \psi_0(y) = \frac{B_0}{a}\sec hay \ \text{ : force free of nonconstant } \alpha\\ &c) \qquad \psi_0(x,y) = -B_0 x\cos ay \, .\end{aligned}$$

By application of the stability criteria deduced in section 2, it can be seen that these equilibria are stable.

Perturbation of the chosen equilibria

Three fourth order differential equations are generated by introducing the above fields into equation 1. We have chosen the integral method to solve our equations which, in this case, proves to be far more powerful than the differential one. The calculation steps are:

- Change of the unknown function: $\hat{\chi} \rightarrow \phi(x,y) = \nabla^2\hat{\chi} = -\Omega_z$. Thus our differential equations get into integro-differential equations.

- Calculation of the Green function of the Laplace operator, ($G(x-x',y,y')$). This funtion results to be convolutive with $\phi(x',y')$.

- Application of a Fourier trasform along x that: removes the inhomogeneity in x in case *c*), reduces the differentiation order in *a*) and *b*) and makes possible the integration in x', by means of the application of the convolution theorem.

  For the cases *a*) and *b*) we get an integral equation:

  $$\left[\frac{\omega^2}{v_A^2} - k^2 f'^2(y)\right]\phi(k,y) = \pi k^2 f'(y) f''(y)\frac{\sinh ky}{\sinh k\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cosh k(\pi - y')\phi(k,y')dy' \ ,$$

  this is a singular integral equation. To regularize it we adapt to our case a method proposed by Sedláček [2].

For the case *c*) instead, we have an integro-differential equation:

$$\frac{\omega^2}{v_A^2}\phi(k,u)+\frac{\partial^2\phi(k,u)}{\partial u^2}+2\frac{\partial}{\partial u}\left[\tanh u\phi(k,u)\right]=$$
$$-\frac{\partial}{\partial u}\left(2\frac{\partial}{\partial k}\left[\cos y\sinh ky\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{k\pi\cosh k(\pi-y'')}{\sinh k\pi}\phi(k,y'')dy''\right]-\right.$$
$$\left.-4\sin y\cosh ky\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{k\pi\cosh k(\pi-y'')}{\sinh k\pi}\phi(k,y'')dy''\right)$$

– The homogeneous part of this equation is a Legendre equation and its solutions are the wellknown Legendre functions.

– The Green function method solves the inhomogeneous equation [4].

– An integro-differential equation remains, with a first order derivative along the transformed coordinate k. The integral nucleus is dyadic.

- The last step consists in the application of the method of the scalar products [4] to the remaining integral equations. The solutions are given in terms of hypergeometric functions.

4. Conclusions

4.1 Solved problems and employed methods:

- Analitic study of the stability of MHD ideal equilibria with traslational symmetry and constant density.

- The novelty we introduce with our work is that the plasma we treat is non-homogeneous and depends on two spatial coordinates.

4.2 New results

- We develop a complete theory, that is, a general methodology to make stability studies in MHD ideal equilibria. This theory provides: a perturbation operator, a stability criterium applicable before the beginning of calculations and an explicit expression for the eigenfunctions.

- We apply our theory to three non-trivial examples, with the consequent resolution of three fourth-order differential equations by means of a generalization and combination of already known methods.

References

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