

COULOMB COLLISIONS IN PLASMA ON MAGNETIC WHITE DWARFS

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Plasma parameters in photospheres of magnetic white dwarfs ($B \sim 10^8$ G, $T \sim 10^4$ K) turned out to be unique as far as Coulomb collisions are concerned. Indeed, the ratio of Larmor radius $r_B = v_T/\omega_B$ to the impact parameter $r_s = e^2/k_B T$ of a scattering to an angle $\theta \sim 1$ in isotropic plasma is small $r_B/r_s = 0.13 \cdot (T/10^4 \text{ K})^{3/2} (B/10^8 \text{ G})^{-1} \ll 1$. To our knowledge, similar values of r_B/r_s were achieved only in low-temperature laboratory experiments with pure electron plasma [1]. Conventional collision terms based on perturbation theory fail to work in this case since effective collisions correspond to strongly disturbed electron trajectories.

In strong magnetic fields the notions of the impact parameter, initial and final velocities as static vectors and the definition of differential cross section based on them lose their meanings. Therefore a novel Boltzmann form of electron-ion and electron-electron collision terms should be obtained. Let us consider a spatially homogeneous ensemble of electrons and motionless ions (or atoms) and describe it by the kinetic equations

$$\frac{\partial f}{\partial t} - [\vec{v}, \vec{\omega}_B] \frac{\partial f}{\partial \vec{v}} - \frac{1}{m} \int \frac{\partial U_{ei}}{\partial \vec{r}'} \frac{\partial f_{ei}}{\partial \vec{v}'} d\vec{r}' - \frac{2}{m} \int \frac{\partial U_{ee}}{\partial \vec{r}'} \frac{\partial f_{ee}}{\partial \vec{v}'_{\text{rel}}} \Big|_{\vec{v}_{\text{cm}} = \vec{v} - \vec{v}'_{\text{rel}}/2} d\vec{v}'_{\text{rel}} d\vec{r}' = 0 \quad (1)$$

$$\frac{\partial f_{ei}}{\partial t'} + \text{div}_{\vec{r}'} \vec{v} f_{ei} - [\vec{v}, \vec{\omega}_B] \frac{\partial f_{ei}}{\partial \vec{v}} - \frac{1}{m} \frac{\partial U_{ei}}{\partial \vec{r}'} \frac{\partial f_{ei}}{\partial \vec{v}} = 0 \quad (2)$$

$$\frac{\partial f_{ee}}{\partial t'} + \text{div}_{\vec{r}'} \vec{v}_{\text{rel}} f_{ee} - [\vec{v}_{\text{rel}}, \vec{\omega}_B] \frac{\partial f_{ee}}{\partial \vec{v}'_{\text{rel}}} - \frac{2}{m} \frac{\partial U_{ee}}{\partial \vec{r}'} \frac{\partial f_{ee}}{\partial \vec{v}'_{\text{rel}}} - [\vec{v}_{\text{cm}}, \vec{\omega}_B] \frac{\partial f_{ee}}{\partial \vec{v}_{\text{cm}}} = 0 \quad (3)$$

for one and two particle distributions $f(\vec{v}, t)$, $f_{ei}(\vec{r}, \vec{v}, t'; t)$, $f_{ee}(\vec{r}, \vec{v}_{\text{rel}}, \vec{v}_{\text{cm}}, t'; t)$, where $U_{ei,ee}(\vec{r})$ are the potentials of electron-ion and electron-electron interactions, \vec{r} is the relative position of an electron in electron-ion or electron-electron pair, relative and center mass velocities \vec{v}_{rel} , \vec{v}_{cm} are chosen as the arguments of f_{ee} to simplify consequent algebra, the additional time variable t' is introduced to exclude explicitly slow variation of the two-particle distributions at collision relaxation time $\sim \nu^{-1}$, $\vec{\omega}_B = e\vec{B}/mc$. Time derivatives $\partial f_{ei}/\partial t'$, $\partial f_{ee}/\partial t'$ in (2), (3) can not be generally neglected in the considered case of rare collisions $\nu \ll \omega_B$ in magnetic field. The former static boundary condition in isotropic plasma for f_{ei} , f_{ee} in the form of uncorrelated colliding particles should be modified to account for the cyclotron gyration of electrons far from the place of a collision

$$f_{ei}(\vec{r}, \vec{v}, t'; t) \Big|_{z(\vec{v}, \vec{z}^\circ) < 0} = n_i f(\hat{F}(t-t')\vec{v}, t) \quad (4)$$

$$f_{ee}(\vec{r}, \vec{v}_{\text{rel}}, \vec{v}_{\text{cm}}, t'; t) \Big|_{z(\vec{v}_{\text{rel}}, \vec{z}^\circ) < 0} = f(\hat{F}(t-t')(\vec{v}_{\text{cm}} + \frac{1}{2}\vec{v}_{\text{rel}}), t) f(\hat{F}(t-t')(\vec{v}_{\text{cm}} - \frac{1}{2}\vec{v}_{\text{rel}}), t). \quad (5)$$

Here $\hat{F}(\tilde{t})$ is the linear operator of rotation $\hat{F}(\tilde{t})\vec{x}^\circ = \vec{x}^\circ \cos \omega_B \tilde{t} + \vec{y}^\circ \sin \omega_B \tilde{t}$, $\hat{F}(\tilde{t})\vec{y}^\circ = -\vec{x}^\circ \sin \omega_B \tilde{t} + \vec{y}^\circ \cos \omega_B \tilde{t}$, $\hat{F}(\tilde{t})\vec{z}^\circ = \vec{z}^\circ$ ($\vec{z}^\circ = \vec{B}/B$). In case of axially symmetrical potentials $U_{ei,ee}(\vec{r}) =$

$U_{ei,ee}(|\vec{r}_\perp|, z)$ and accepted boundary conditions (4), (5) the distributions f_{ei} and f_{ee} acquire a significant feature of rotation both in coordinate and velocity phase spaces. This symmetry leads to the following coupling of temporal and spatial derivatives in addition to the kinetic equations (2), (3)

$$\frac{\partial f_{ei}}{\partial t'} - [\vec{r}, \vec{\omega}_B] \frac{\partial f_{ei}}{\partial \vec{r}} - [\vec{v}, \vec{\omega}_B] \frac{\partial f_{ei}}{\partial \vec{v}} = 0 \quad (6)$$

$$\frac{\partial f_{ee}}{\partial t'} - [\vec{r}, \vec{\omega}_B] \frac{\partial f_{ee}}{\partial \vec{r}} - [\vec{v}_{\text{rel}}, \vec{\omega}_B] \frac{\partial f_{ee}}{\partial \vec{v}_{\text{rel}}} - [\vec{v}_{\text{cm}}, \vec{\omega}_B] \frac{\partial f_{ee}}{\partial \vec{v}_{\text{cm}}} = 0 \quad (7)$$

These relations in combination with (2) and (3) provide an opportunity to present the collision terms as the electron fluxes through distant surfaces surrounding the scattering center and rotating at cyclotron frequency

$$I_{\text{st}}^{ei}(\vec{v}, t) = \oint_{\vec{r}_s \in S} f_{ei}(\vec{r}_s, \vec{v}, t' = t; t) (\vec{v} - [\vec{\omega}_B, \vec{r}_s], d\vec{S}) \quad (8)$$

$$I_{\text{st}}^{ee}(\vec{v}, t) = \int d\vec{v}_{\text{rel}} \oint_{\vec{r}_s \in S} f_{ee}(\vec{r}_s, \vec{v}_{\text{rel}}, \vec{v} - \frac{1}{2}\vec{v}_{\text{rel}}, t' = t; t) (\vec{v}_{\text{rel}} - [\vec{\omega}_B, \vec{r}_s], d\vec{S}) \quad (9)$$

For any surface S the differential forms $(\vec{v}' - [\vec{\omega}_B, \vec{r}_s], d\vec{S})$ (\vec{v}' stands for \vec{v} or \vec{v}_{rel}) are equal to $\text{sign}((\vec{v}' - [\vec{\omega}_B, \vec{r}_s], \vec{n}_s)) |v'_z R dR d\Phi_0|$ where $\vec{R} = x_s \vec{x}^\circ + y_s \vec{y}^\circ - [\vec{v}', \vec{\omega}_B] / \omega_B^2$ is the position of cyclotron gyration center (of the total or relative motion) in transverse plane and Φ_0 is the angle between the vector \vec{R} and the Larmor radius $[\vec{v}'(t_0), \vec{\omega}_B] / \omega_B^2$ at the moment t_0 when an electron would cross the plane $z = 0$ along the continued unperturbed helical trajectory $(x_s = Rv'_y/v'_\perp \cos(\Phi_0 + \omega_B z_s(x_s, y_s)/v'_z) - Rv'_x/v'_\perp \sin(\Phi_0 + \omega_B z_s(x_s, y_s)/v'_z) + v'_y, y_s = -Rv'_x/v'_\perp \cos(\Phi_0 + \omega_B z_s(x_s, y_s)/v'_z) - Rv'_y/v'_\perp \sin(\Phi_0 + \omega_B z_s(x_s, y_s)/v'_z) - v'_x)$. The variables R, Φ_0 are identical to b, Θ which were introduced in [2] to parametrize test particle helices. Finally, to obtain the desired Boltzmann form the functions f_{ei} and f_{ee} in (8), (9) should be expressed via $f(\vec{v}, t)$. The constancy of f_{ei} and f_{ee} along the collision trajectory coupled with the boundary conditions (4), (5) gives the desired expressions for incoming and outgoing electrons and the collision terms acquire the final form

$$I_{\text{st}}^{ei} = \int_0^\infty dR \int_0^{2\pi} d\Phi_0 R |v_z| [f(\vec{v}_{ei}^*(-\vec{v}, R, -\Phi_0), t) - f(\vec{v}, t)] \quad (10)$$

$$I_{\text{st}}^{ee} = \int d\vec{v}_{\text{rel}} \int_0^\infty dR \int_0^{2\pi} d\Phi_0 R |v_{\text{rel}z}| [f(\vec{v} - \frac{1}{2}(\vec{v}_{\text{rel}} - \vec{v}_{ee}^*(-\vec{v}_{\text{rel}}, R, -\Phi_0)), t) \times \quad (11)$$

$$\times f(\vec{v} - \frac{1}{2}(\vec{v}_{\text{rel}} + \vec{v}_{ee}^*(-\vec{v}_{\text{rel}}, R, -\Phi_0)), t) - f(\vec{v}, t)f(\vec{v} - \vec{v}_{\text{rel}}, t)]$$

where $\vec{v}_{ei,ee}^*(\vec{u}, R, \Phi) = -\vec{z}^\circ u \cos \theta_{ei,ee} - ([\vec{z}^\circ, \vec{u}] \sin(-\Delta\phi_{ei,ee}) + [[\vec{z}^\circ, \vec{u}], \vec{z}^\circ] \cos(-\Delta\phi_{ei,ee})) \times \sin \theta_{ei,ee} / \sin \theta$. The functions $\theta_{ei,ee}(\vec{u}, R, \Phi)$ and $\Delta\phi_{ei,ee}(\vec{u}, R, \Phi)$ represent the final pitch angle and the shift in the initial phase of cyclotron gyration for an electron that starts with initial pitch angle $\theta = \arccos((\vec{u}, \vec{B}')/uB)$ and helix (R, Φ) and collides with the motionless potential U_{ei} or $2U_{ee}$ but in the inverse magnetic field $\vec{B}' = -\vec{B}$.

The obtained collision terms for helical initial trajectories are valid at any ratio of electron Larmor radius to the distance of interaction with a scattering center. In the limit $B \rightarrow 0$ they naturally reduce to the usual Boltzmann term where the impact parameter $\vec{p} = v'_z [[\vec{z}^\circ, \vec{v}'] \vec{v}'] (\Phi_0 - \pi) / v'^2 + [\vec{z}^\circ, \vec{v}'] (R - v'_\perp / \omega_B) / v'_\perp$ and the region of parameters $|\omega_B R - v'_\perp| \ll v'_\perp, |\Phi_0 - \pi| \ll$

1 yields the major contribution to the integrals. The redefined cross section $\sigma(v, \vec{n}, \vec{n}^*) = R|v'_z|/v' |\partial(\vec{n}^*)/\partial(R, \Phi_0)|^{-1}$ allows one to write collision terms in standard forms of integrals over the direction $\vec{n}^* = \vec{v}_{ei,ee}^*/v_{ei,ee}^*$ ($\vec{n} = \vec{v}'/v'$).

To obtain the cross section in strong magnetic field let us consider classical motion of an electron in homogeneous magnetic field $\vec{B} = B\vec{z}^{\circ}$ and in Coulomb field of a motionless ion with a charge $\pm Ze$ ($e > 0, Z > 0$). The initial dynamic equations allow introduction of such dimensionless coordinate $\vec{r}' = \vec{r}/L_0$ ($L_0 = (Zmc^2/B^2)^{1/3}$) and time $\tau = \omega_B t$ that the only parameter of equivalent system

$$\frac{d^2\vec{r}'}{d\tau^2} = -\left[\frac{d\vec{r}'}{d\tau}, \vec{z}^{\circ}\right] \mp \frac{\vec{r}'}{r'^3} \quad (12)$$

is the sign of ion. Firstly, it means that a change of magnetic field strength causes only scale transformation of trajectories in the phase space of initial system. Secondly, the bifurcation in phase space takes place only due to a switch of the charge sign of the ion. Hereafter, only the dimensionless form of all values will be used.

The types of motions described by (12) can be roughly classified via the total dimensionless mechanic energy E . In case of large energies $|E| \gg 1$ (either positive or negative) the nearly Keplerian orbits can be observed. The hyperbolic loop connects two helixes of cyclotron gyration if impact parameter $R \ll r_B$. The elliptic ones undergo Larmor precession. The influence of magnetic field is the most striking for electron states near the boundary of finite and infinite motions $|E| \ll 1$. The finite classical trajectories present an analogue to a hydrogen atom in a superstrong magnetic field $B \geq 10^9$ G [3]. The electron motion in transverse plane resembles cyclotron gyration combined with a slow azimuth drift. Along the magnetic field, the electron oscillates in a one-dimensional Coulomb potential.

Let us focus on the electron-proton collisions in the region of parameters $E \ll 1$ realized for electrons in photospheres of magnetic white dwarfs. Zheleznyakov et al. [4] proposed to divide all the collisions in two groups. For the distant ones ($R > p_{\max}$) the transverse kinetic energy is the adiabatic invariant and thus it is conserved with high accuracy. The effective collisions corresponding to $R < p_{\max}$ are characterized by significant change of pitch angle. The estimate $p_{\max} \sim 1$ was found via the equality of cyclotron period to the characteristic time of electric field variation along the corresponding trajectory with $R \sim p_{\max}$. It was shown that p_{\max} falls far below $r_s = 1/v^2$ if $E \ll 1$ so that the acceleration of an electron during an effective collision must be considered. The latter causes a significant independence of p_{\max} on the initial electron velocity and provides $p_{\max} > r_B = v$. The essential feature of effective collisions lies in a quasi-bound form of trajectories because the probability for an electron to leave an ion is very small once it's transverse energy has been changed by the value ~ 1 at the first encounter with the ion which exceeds it's initial kinetic energy many times.

The more detailed analysis reveals a logarithmic dependence of p_{\max} on E . In conformity with (12), the square of the final transverse velocity $v_{\perp}^2 = |v_x - iv_y - v_c|$ where $v_c = \int_{-\infty}^{+\infty} dt \exp(it)(x(t) - iy(t))/r^3(t)$ and v_x, v_y are the components of oscillating initial

electron velocity at arbitrary moment $t = 0$. Taking into account the acceleration of an electron along the magnetic field

$$v_c = i \exp(i\Phi_0) \frac{v_x - iv_y}{v_\perp} \frac{2^{1/10} \sqrt{5 - \sqrt{5}} \Gamma(4/5)}{5^{1/5} R^{1/5}} \exp\left(-R^{3/2} \frac{\sqrt{\pi} \Gamma(1/4)}{6\sqrt{2} \Gamma(3/4)}\right) \quad (13)$$

for $1 \leq R \ll 1/v^2$ was found by substitutions $z(t) = \int_0^t \sqrt{v_z^2 + 2/(R^2 + z^2(t))} dt \approx \sqrt{2} R^{-1/2} \int_0^t (1 + (z(t)/R)^2)^{-1/4} dt$ and $x(t) - iy(t) = (v_x - iv_y)/v_\perp \cdot R \exp(i\pi/2 + i\Phi_0)$. Since the obtained estimate of v_c is a rapidly decreasing function of the impact parameter R , the value

$$p_{\max} = \left(\frac{\sqrt{\pi} \Gamma(1/4)}{6\sqrt{2} \Gamma(3/4)} \right)^{-2/3} \ln^{2/3} v^{-1} \quad (14)$$

can be found with logarithmic accuracy immediately from the first order evaluation (13) by putting $v_c(R)$ equal to v . For the very slow electrons $v \ll 1$ the interval of impact parameters ΔR separating effective and ineffective collisions becomes more and more narrow so that $\Delta R/p_{\max} \sim \ln^{-1} v^{-1} \ll 1$.

Quasi-trapped form of effective collision trajectories leaves little to analytical investigation. However it seems natural to assume a statistical independence of the final velocity on the initial one because of rapid variation of the final pitch angle and the shift in the initial phase of the cyclotron gyration with the impact parameter in the region $R \leq p_{\max}$. This hypothesis means the form of differential cross section to be $\sigma(v, \vec{n}, \vec{n}') = \tilde{\sigma}_0(v) w(\vec{n}) w'(\vec{n}')$. In the considered case of a small Larmor radii ($v \ll p_{\max}$) and a sharp separation of the effective and ineffective collisions the proposed supposition yields the only $\sigma(v, \vec{n}, \vec{n}') = \sigma_0(v) |\cos \theta \cos \theta'| / 2\pi$ where $\sigma_0(v) = \pi p_{\max}^2(v)$. It is worth noting that $\sigma(v, \vec{n}, \vec{n}')$ corresponds to equipartition of a colliding electron over all Landau levels allowed by energy conservation because $\partial f_\perp(\vec{v}_\perp) / \partial t = \int dv_z I_{st}^{ei} = n_i \sigma_0(v_0) \cos \theta_0 / (2\pi v_0) |_{v_\perp < v_0}$ is independent on v_\perp for a single electron distribution $f(\vec{v}) = \delta(\vec{v} - \vec{v}_0)$.

The carried out analysis provides the transport frequency $\nu_{\text{eff}} = 4\sqrt{2\pi}/5 n_i p_{\max}^2 v_T$ responsible for the equipartition rate $dT_\perp/dt = \nu_{\text{eff}}(T_\parallel - T_\perp)$. It plays the essential role in cyclotron radiation transfer in photospheres of magnetic white dwarfs.

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