GEOMETRIC STRUCTURES IN TOROIDAL PLASMA CONTAINMENT

G. Fuchs

Institut für Plasmaphysik Assiciation–Euratom, Trilateral Euregio Cluster
Forschungszentrum Jülich GmbH D–52425 Jülich Germany

Abstract

The structure of a tokamak magnetic field is that of a Hamiltonian flow. This feature can be used to give the field a geometrical meaning. This is accomplished by using the tools of modern differential geometry namely that of k–forms. The geometrical structures are furthermore applied to those plasma diagnostics, which give line integrated data and they lead there to very visual structures and evaluation methods.

1. Introduction

Toroidal plasma containment is still being investigated using mainly vector notation. One can, however, as well use tensor notation or k–forms (skew symmetric covariant tensors). Especially with k–forms there are accompanying geometrical structures, making things – after being acquainted to the subject – very visual. Those who have no experience with k–forms find a visual introduction in [1] pp. 60..120, [2] gives many graphs and physical examples, [3] and [4] are more mathematical.

Moreover, k–forms can provide links between the measurements and their interpretation, this is shown using the example of a vector–tomographic problem, namely the interpretation of Faraday rotation data.

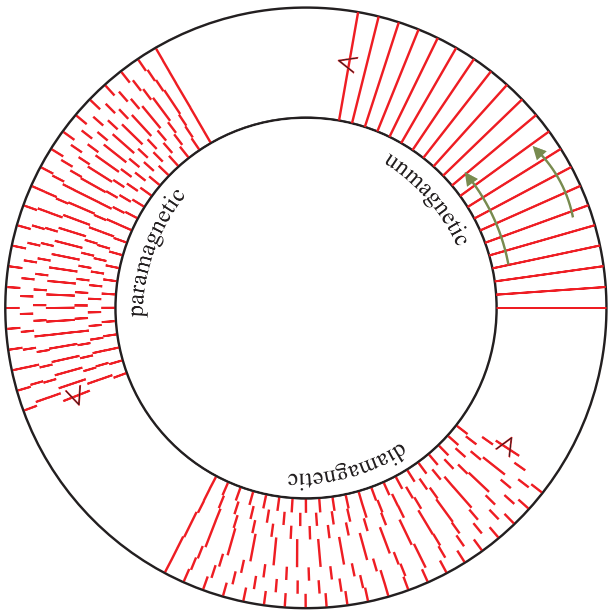

Figure 1. The figure shows a schematic of both, the 1-form of the toroidal magnetic field or the vector potential to create the 2-form of the poloidal one. The vectors are indicated as well.

2. Magnetic fields in toroidal symmetry

Here k–forms are used to visualize the magnetic field structure of tokamaks. We start with the toroidal field, which is simply a 1–form shown in Fig. 1 as projection onto the midplane.

We have

$$\tilde{H}_t = F\,\mathrm{d}\varphi = \frac{1}{\mu_0} \star \tilde{B}_t \tag{1}$$

Because of toroidal symmetry F is a function of the coordinates in a poloidal plane, d the differential operator (equivalent to grad, curl and div in vectoranalysis), φ is the toroidal angle and $\star$ the hodge star operator linking, together with μ_0, the magnetic field to the inductive flux. $\star$ transforms a k–form in n–dim space to a (n-k)–form, i.e. for n=3 in our case a 1–form to a 2–form. Notice that the vacuum field only gives undisrupted planes. The diamagnetic current density is the 2–form

$$\tilde{j}_{dia} = \mathrm{d}(F\mathrm{d}\varphi) = \mathrm{d}F \wedge \mathrm{d}\varphi, \tag{2}$$

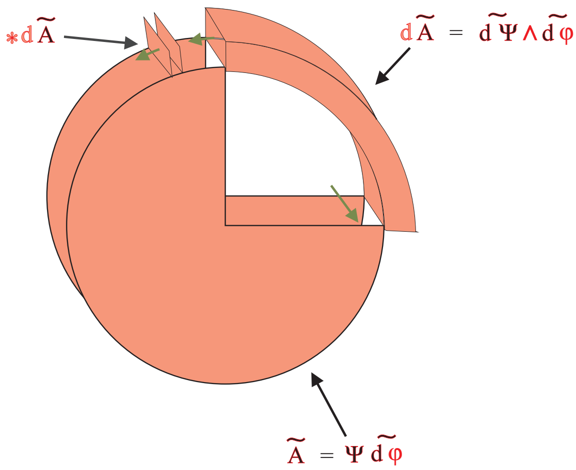

Figure 2. The figure shows a schematic of the 1-form of the toroidal vector potential creating the poloidal magnetic flux $\tilde{B}$ and it's 1-form. The same geometry does apply, starting from the toroidal field (cf.Fig. 1), for the diamagnetic current density.

where $\wedge$ is the outer product (see e.g. [2] p.31). Because of dd $= 0$ the flux j_{dia} is conserved, which implies that it is a Hamiltonian flux.

For the poloidal field and the toroidal current flux we start from the vector potential, which is in toroidal direction and we notice that for the same reason as before for $\tilde{j}_{dia}$ now $\tilde{H}_p = 1/\mu_0 \star \tilde{B}_p$ is a Hamiltonian flow.

$$\begin{aligned} \tilde{A} &= \Psi\,\mathrm{d}\varphi \\ \tilde{B}_p &= \mathrm{d}(\Psi\mathrm{d}\varphi) = \mathrm{d}\Psi \wedge \mathrm{d}\varphi \\ \tilde{j}_t &= \mathrm{d}\tilde{H}_p = \frac{1}{\mu_0}\mathrm{d} \star \tilde{B}_p \end{aligned} \tag{3}$$

To derive the condition for equilibrium, as done in [5] we start from the Lagrangian, it is the Legendre transform of the total energy and it's variation has to vanish.

$$\begin{aligned} L(\frac{\partial\Psi}{\partial x^i}, x^i) = \frac{\mu_0}{2} < H_p|H_p > -\frac{\mu_0}{2} < H_t|H_t > -p \\ \delta \int L\,dV = 0 \end{aligned} \tag{4}$$

$< | >$ is the inner product and we have made use of $< H|H >= \star(H \wedge \star H)$ (cf. [2] p.160). The Grad–Shafranov equation is the Euler–Lagrange equation of this variational problem, whereby the operators $\frac{\partial}{\partial t}$ and dt known form classical mechanics have to be replaced by $\star\mathrm{d}\star$ (the adjoint operator to d) and by dV.

Here is the coordinate independent form of it

$$\star \mathrm{d} \star \frac{1}{R^2}\mathrm{d}\Psi + \frac{\mu_0}{R^2}F\mathrm{d}F + \mathrm{d}p = 0 \tag{5}$$

To give the result in coordinates which have toroidal symmetry and are arbitrary otherwise we introduce a metric tensor of the kind

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & R^2 \end{pmatrix} \tag{6}$$

Introducing action angle variables Φ (the best choice is the toroidal flux within an isocontour of Ψ), ϑ in the poloidal plane, the equation for equilibrium is[1]:

$$g_{22}\frac{\partial^2\Psi}{\partial\Phi} + \frac{\partial g_{22}}{\partial\Phi} - \frac{\partial g_{21}}{\partial\vartheta} + \frac{q\mu_0 F}{R^2}\frac{\partial F}{\partial\Phi} + q\frac{\partial p}{\partial\Phi} = 0 \tag{7}$$

where q is the safety factor and it has been assumed, that the Liouville theorem holds, which gives det $g = 1$.

3. Vector tomography on k-forms

Many measurements on tokamaks give line integrated data i.e. sinograms, which are to be inverted in order to obtain local pictures i.e. tomograms. In many cases there are insufficient data e.g. there is only one angle of observation. To make up for this deficiency one must make assumptions which ought to be physical. A vector-tomographic problem is of the kind, that line integrals of a 1–form are known for different impact parameters and angles and the 1–form is sought. The data are in this case insufficient to do the inversion. However, if the 1–form is exact, i.e. it is the differential of a function or if it constitutes a Hamiltonian flow in 2–dim, then the solution is possible. The retrieval of the Ψ function from Faraday rotation is of this kind. The case has been investigated in vector notation in [6].

$$\alpha = c_f \int_{\mathcal{C}} n\tilde{H}\, dl \tag{8}$$

α is the Faraday rotation angle, c_f a constant, n the plasma density and dl a line element along a curve $\mathcal{C}$. Because n can be assumed to depend on the flux surface label only i.e. $n(\Psi)$, we can introduce a function $\Xi(\Psi)$ such that $n = d\,\Xi/d\Psi$ and, following equs(3), write

$$\alpha = c_f \int_{\mathcal{C}} \star(\mathrm{d}\,\Xi \wedge \mathrm{d}\,\varphi)\, dl. \tag{9}$$

[1] D. Palumbo has arrived at the same result doing the calculations in conventional vector manner. The author has used a Maple program based on tensors to perform the calculations.

There is now a visual explanation for α: it is the number of times the line element pierces the 1–form $\star(\mathrm{d}\,\Xi\wedge\mathrm{d}\,\varphi)$ along $\mathcal{C}$; this is depicted in Fig. 3. Following [6], the probing beams can be created as the line element contact bundles of a function X, in which case we get

$$\begin{aligned}\alpha &= c_f\int_{\mathcal{C}} n\star(\mathrm{d}X\wedge\mathrm{d}\Xi\wedge\mathrm{d}\varphi)\,dl\\ &= \mathcal{R}n\star(\mathrm{d}X\wedge\mathrm{d}\Xi\wedge\mathrm{d}\varphi)\end{aligned} \tag{10}$$

where $\mathcal{R}$ is the Radon operator (cf. V.Pickalov et al. this issue).

The interesting point is, that the vector-tomographic problem has been reduced to a scalar one. For inversion it is referred to the vast literature in this field (see e.g. citations in [6]).

References

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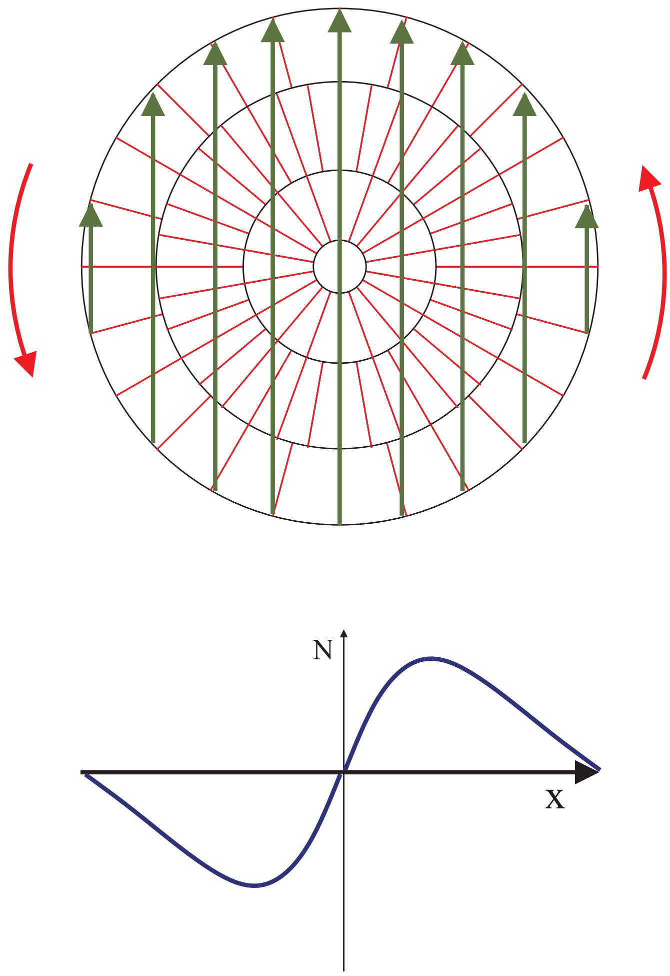

Figure 3. The measurement of the Faraday rotation is explained, it gives the number of times a set of vectors pierces a 1–form of the poloidal magnetic field which has been scaled by the density function. The result is sketched on the bottom.