SOLUTION OF FULL WAVE EQUATION FOR GLOBAL MODES IN LOW ASPECT RATIO TOKAMAKS WITH NON-CIRCULAR CROSS-SECTION

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Abstract

In this paper the wave equation for low aspect ratio (spherical) tokamaks with non-circular cross-section is properly formulated and solved for the waves in the Alfven frequency range. The problem is formulated in terms of the vector and scalar potential (**A**, **Φ**). The equilibrium configuration is reconstructed from experimental results and equatorial computations presented by Wilson [3] for START [2].

The wave equation is solved by the aid of a numerical code adapted for the present problem, based on the general $2\frac{1}{2}$D finite element solver proposed by Sewell [1]. With the definitions $f_i(\theta,\rho) = F_i(-\theta,\rho)$, $(f_i, F_i = A_j, \Phi;\ \ j = \rho,\theta,\phi)$, our code solves simultaneously 16 second order partial differential equations.

1. Explicit formulation of the wave equation

The wave equations, with the dielectric tensor operator described above, can be casted into the following form, suitable for numerical integration ($\boldsymbol{\mathcal{A}} \equiv i\omega\mathbf{A}$ $\mathbf{A}, \Phi$ – the e.m. potentials, $\partial_\theta \equiv\ _{,\theta} \equiv \partial/\partial\theta$):

$$\partial_\theta \mathbf{L} + \partial_\rho \mathbf{M} = \hat{\mathsf{F}} \cdot \mathbf{U} - \mathbf{H}, \tag{1}$$

$$\mathbf{U} \equiv \begin{bmatrix} \mathcal{A}_\rho \\ \mathcal{A}_\theta \\ \mathcal{A}_\phi \\ \Phi \end{bmatrix}, \quad \mathbf{H} \equiv -\frac{\rho^2}{k_0^2} \begin{bmatrix} \mathcal{J}_\rho \\ \mathcal{J}_\theta \\ \mathcal{J}_\phi \\ 0 \end{bmatrix}, \ \mathbf{L} = \begin{bmatrix} \mathcal{A}_{\rho,\theta}/k_0^2 \\ \mathcal{A}_{\theta,\theta}/k_0^2 \\ \mathcal{A}_{\phi,\theta}/k_0^2 \\ \mathbf{L}_4 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \rho^2 \mathcal{A}_{\rho,\rho}/k_0^2 \\ \rho^2 \mathcal{A}_{\theta,\rho}/k_0^2 \\ \rho^2 \mathcal{A}_{\phi,\rho}/k_0^2 \\ \mathbf{M}_4 \end{bmatrix}, \tag{2}$$

$$\mathbf{L}_4 \equiv \Omega_{1\theta\rho}\, \rho\Phi_{,\rho} + \Omega_{1\theta\theta}\, \Phi_{,\theta} + \Omega_{1\theta\phi}\, \rho h_\phi{}^{-1}\partial_\phi\Phi - \rho\varepsilon^0_{\theta\rho}\mathcal{A}_\rho - \rho\varepsilon^0_{\theta\theta}\mathcal{A}_\theta - \rho\varepsilon^0_{\theta\phi}\mathcal{A}_\phi -$$
$$9\varepsilon^\theta_{\theta\rho}\mathcal{A}_{\rho,\theta} - \varepsilon^\theta_{\theta\phi}\mathcal{A}_{\phi,\theta} - \rho\varepsilon^\rho_{\theta\theta}\mathcal{A}_{\theta,\rho} - \rho\varepsilon^\rho_{\theta\phi}\mathcal{A}_{\phi,\rho}\,, \tag{3}$$

$$\mathbf{M}_4 \equiv \rho\, \Omega_{1\rho\rho}\, \rho\Phi_{,\rho} + \rho\, \Omega_{1\rho\theta}\, \Phi_{,\theta} + \rho\, \Omega_{1\rho\phi}\, \rho h_\phi{}^{-1}\partial_\phi\Phi - \rho^2\varepsilon^0_{\rho\rho}\mathcal{A}_\rho - \rho^2\varepsilon^0_{\rho\theta}\mathcal{A}_\theta - \rho^2\varepsilon^0_{\rho\phi}\mathcal{A}_\phi -$$
$$-\ \rho\, \varepsilon^\theta_{\rho\rho}\mathcal{A}_{\rho,\theta} - \rho\, \varepsilon^\theta_{\rho\phi}\mathcal{A}_{\phi,\theta} - \rho^2\varepsilon^\rho_{\rho\theta}\mathcal{A}_{\theta,\rho} - \rho^2\varepsilon^\rho_{\rho\phi}\mathcal{A}_{\phi,\rho}\,. \tag{4}$$

The matrix-operator $\hat{\mathsf{F}}$, in eq.(1), has the structure: $\hat{\mathsf{F}} \equiv \mathsf{F}^0 + \mathsf{F}^1 + \mathsf{F}^\theta \partial_\theta + \mathsf{F}^\rho \partial_\rho$, with

$$\mathsf{F}^0 \equiv \frac{1}{k_0^2} \begin{bmatrix} \chi - \rho^2 g_{,\rho\rho} & -\rho g_{,\theta\rho} + 2g_{,\theta} & 2\rho^2 g_{,\rho} {h_\phi}^{-1}\partial_\phi & 0 \\ -\rho g_{,\theta\rho} & \chi - \rho g_{,\rho} - g_{,\theta\theta} & 2\rho g_{,\theta} {h_\phi}^{-1}\partial_\phi & 0 \\ -2\rho^2 g_{,\rho} {h_\phi}^{-1}\partial_\phi & -2\rho g_{,\theta} {h_\phi}^{-1}\partial_\phi & \chi - 1 - \rho g_{,\rho} - g_{,\theta\theta} - \rho^2 g_{,\rho\rho} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \tag{5}$$

$$\mathsf{F}^1 \equiv \rho \begin{bmatrix} -\rho\varepsilon^0_{\rho\rho} & -\rho\varepsilon^0_{\rho\theta} & -\rho\varepsilon^0_{\rho\phi} & \varepsilon^\phi_{\rho\phi} {h_\phi}^{-1}\partial_\phi \\ -\rho\varepsilon^0_{\theta\rho} & -\rho\varepsilon^0_{\theta\theta} & -\rho\varepsilon^0_{\theta\phi} & \varepsilon^\phi_{\theta\phi} {h_\phi}^{-1}\partial_\phi \\ -\rho\varepsilon^0_{\phi\rho} & -\rho\varepsilon^0_{\phi\theta} & -\rho\varepsilon^0_{\phi\phi} & \varepsilon^\phi_{\phi\phi} {h_\phi}^{-1}\partial_\phi \\ \mathsf{F}^1_{41} & \mathsf{F}^1_{42} & \mathsf{F}^1_{43} & \mathsf{F}^1_{44} \end{bmatrix} , \tag{6}$$

$$\mathsf{F}^\theta \equiv \begin{bmatrix} -\rho\varepsilon^\theta_{\rho\rho} - g_{,\theta}/k_0^2 & 2/k_0^2 & -\rho\varepsilon^\theta_{\rho\phi} & \varepsilon^\phi_{\rho\theta} \\ -\rho\varepsilon^\theta_{\theta\rho} - 2/k_0^2 & -g_{,\theta}/k_0^2 & -\rho\varepsilon^\theta_{\theta\phi} & \varepsilon^\phi_{\theta\theta} \\ -\rho\varepsilon^\theta_{\phi\rho} & 0 & -\rho\varepsilon^\theta_{\phi\phi} - g_{,\theta}/k_0^2 & \varepsilon^\phi_{\phi\theta} \\ \mathsf{F}^\theta_{41} & 0 & \mathsf{F}^\theta_{43} & \mathsf{F}^\theta_{44} \end{bmatrix} , \tag{7}$$

$$\mathsf{F}^\rho \equiv \rho \begin{bmatrix} (1 - \rho g_{,\rho})/k_0^2 & -\rho\varepsilon^\rho_{\rho\theta} & -\rho\varepsilon^\rho_{\rho\phi} & \varepsilon^\phi_{\rho\rho} \\ 0 & -\rho\varepsilon^\rho_{\theta\theta} + (1 - \rho g_{,\rho})/k_0^2 & -\rho\varepsilon^\rho_{\theta\phi} & \varepsilon^\phi_{\theta\rho} \\ 0 & -\rho\varepsilon^\rho_{\phi\theta} & -\rho\varepsilon^\rho_{\phi\phi} + (1 - \rho g_{,\rho})/k_0^2 & \varepsilon^\phi_{\phi\rho} \\ 0 & \mathsf{F}^\rho_{42} & \mathsf{F}^\rho_{43} & \mathsf{F}^\rho_{44} \end{bmatrix} . \tag{8}$$

In these matrix expressions, the following notations have been used:

$$\chi \equiv 1 - \rho^2 ({h_\phi}^{-1}\partial_\phi)^2 , \quad \varepsilon^\phi \equiv \rho\, \Omega_1 , \ \mathsf{F}^1_{41} = (\rho g_{,\rho} - 1)\varepsilon^0_{\rho\rho} + g_{,\theta}\varepsilon^0_{\theta\rho} + \varepsilon^0_{\phi\rho}\rho {h_\phi}^{-1}\partial_\phi , \tag{9}$$

$$\mathsf{F}^1_{42} = (\rho g_{,\rho} - 1)\varepsilon^0_{\rho\theta} + g_{,\theta}\varepsilon^0_{\theta\theta} + \varepsilon^0_{\phi\theta}\rho {h_\phi}^{-1}\partial_\phi , \ \mathsf{F}^1_{43} = (\rho g_{,\rho} - 1)\varepsilon^0_{\rho\phi} + g_{,\theta}\varepsilon^0_{\theta\phi} + \varepsilon^0_{\phi\phi}\rho {h_\phi}^{-1}\partial_\phi , \tag{10}$$

$$\mathsf{F}^1_{44} = -(\rho g_{,\rho} - 1)\, \Omega_{1\rho\phi} {h_\phi}^{-1}\partial_\phi - g_{,\theta}\, \Omega_{1\theta\phi} {h_\phi}^{-1}\partial_\phi - \rho {h_\phi}^{-1}\partial_\phi\, \Omega_{1\phi\phi} {h_\phi}^{-1}\partial_\phi , \tag{11}$$

$$\mathsf{F}^\theta_{41} = (\rho g_{,\rho} - 1)\varepsilon^\theta_{\rho\rho} + g_{,\theta}\varepsilon^\theta_{\theta\rho} + \varepsilon^\theta_{\phi\rho}\rho {h_\phi}^{-1}\partial_\phi , \ \mathsf{F}^\theta_{43} = (\rho g_{,\rho} - 1)\varepsilon^\theta_{\rho\phi} + g_{,\theta}\varepsilon^\theta_{\theta\phi} + \varepsilon^\theta_{\phi\phi}\rho {h_\phi}^{-1}\partial_\phi , \tag{12}$$

$$\mathsf{F}^\theta_{44} = -(\rho g_{,\rho} - 1)\, \Omega_{1\rho\theta} - g_{,\theta}\, \Omega_{1\theta\theta} - \rho {h_\phi}^{-1}\partial_\phi\, \Omega_{1\phi\theta} , \tag{13}$$

$$\mathsf{F}^\rho_{42} = (\rho g_{,\rho} - 1)\varepsilon^\rho_{\rho\theta} + g_{,\theta}\varepsilon^\rho_{\theta\theta} + \varepsilon^\rho_{\phi\theta}\rho {h_\phi}^{-1}\partial_\phi , \ \mathsf{F}^\rho_{43} = (\rho g_{,\rho} - 1)\varepsilon^\rho_{\rho\phi} + g_{,\theta}\varepsilon^\rho_{\theta\phi} + \varepsilon^\rho_{\phi\phi}\rho {h_\phi}^{-1}\partial_\phi , \tag{14}$$

$$\mathsf{F}^\rho_{44} = -(\rho g_{,\rho} - 1)\, \Omega_{1\rho\rho} - g_{,\theta}\, \Omega_{1\theta\rho} - \rho\rho {h_\phi}^{-1}\partial_\phi\, \Omega_{1\phi\rho} . \tag{15}$$

Notice that in (2) we used the notation

$$\mathcal{J} \equiv \mu_0 i c k_0 \mathbf{j}_{antenna} = \mu_0 i c k_0 \delta(\rho - \rho_a(\theta)) \left\{ -\mathbf{e}_\theta \frac{in}{h_\phi} + \mathbf{e}_\phi \frac{im}{h_\theta} \right\} \mathcal{C} , \tag{16}$$

where $\mathcal{C}$ is an arbitrary constant, m and n are the poloidal and toroidal wave numbers and δ stands for Dirac function used for the localization of the antenna.

2. Numerical Solution

For the numerical solution of the wave equation we used the code PDE2D [1] — a general purpose finite element program which solves systems of nonlinear time-dependent, nonlinear steady-state and linear eigenvalue partial differential equations, in general 2D regions. The solver uses a Galerkin finite element method, with isoparametric triangular elements of up to 4th degree. For the illustrative case considered here $(m = 3, n = 1)$, the number of finite elements used was $N_{\triangle} = 4 * (N_\theta - 1) * (N_\rho - 1) = 4240$

3. Concluding Remarks

We developed a rigorous formalism for the wave equation, appropriate for the study of propagation, absorption and mode conversion of externally launched waves in spherical tokamaks with non-circular cross-section. The formalism includes a rigorous regularity, boundary, gauge and periodicity conditions suitable for the exact solution of the equation.

By this formalism, we numerically solved the problem for an ohmic START-like ($R/a \approx 1.4$) plasma, surrouanded by a sheet-current antenna launching fast waves to the plasma. The powerful finite element code used provides reliable results representing a generalization of those obtained under various simplifying assumptions.

As a new and basic result we mention the clear demonstration of significant wave absorption and conversion which for illustrative antenna parameters used here are strongly localized in the almost equatorial high field side (Figs.1–2).

References

[1] Sewell, G.: Advances in Engineering Software **17**, 105 (1993)

[2] Sykes, A.: Plasma Phys. and Contr. Fusion **36**, B93 (1994)

[3] Wilson, H.R.: Euratom-UKAEA Report, Culham, Abingdon, UK. (1994)

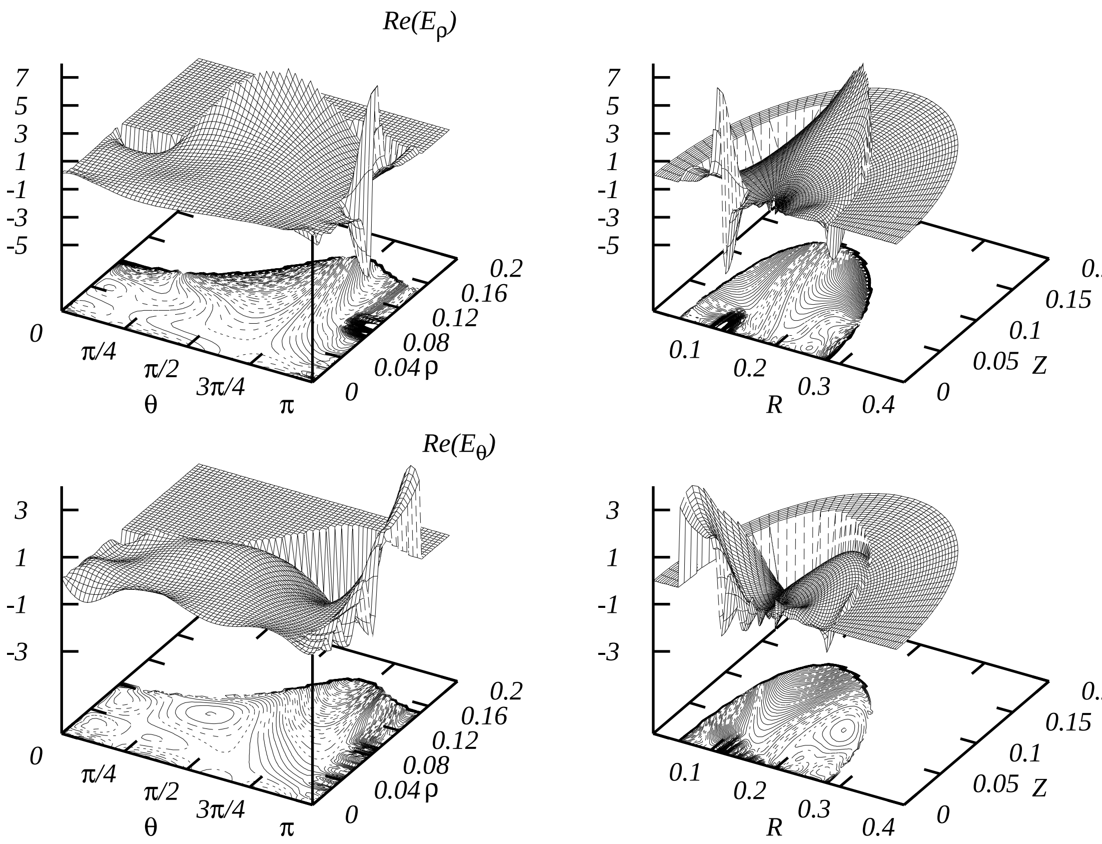

Figure 1. Solutions of the wave equation for $m = 3,\ n = 1,\ \omega = 1.5 * 10^7\ s^{-1}$.

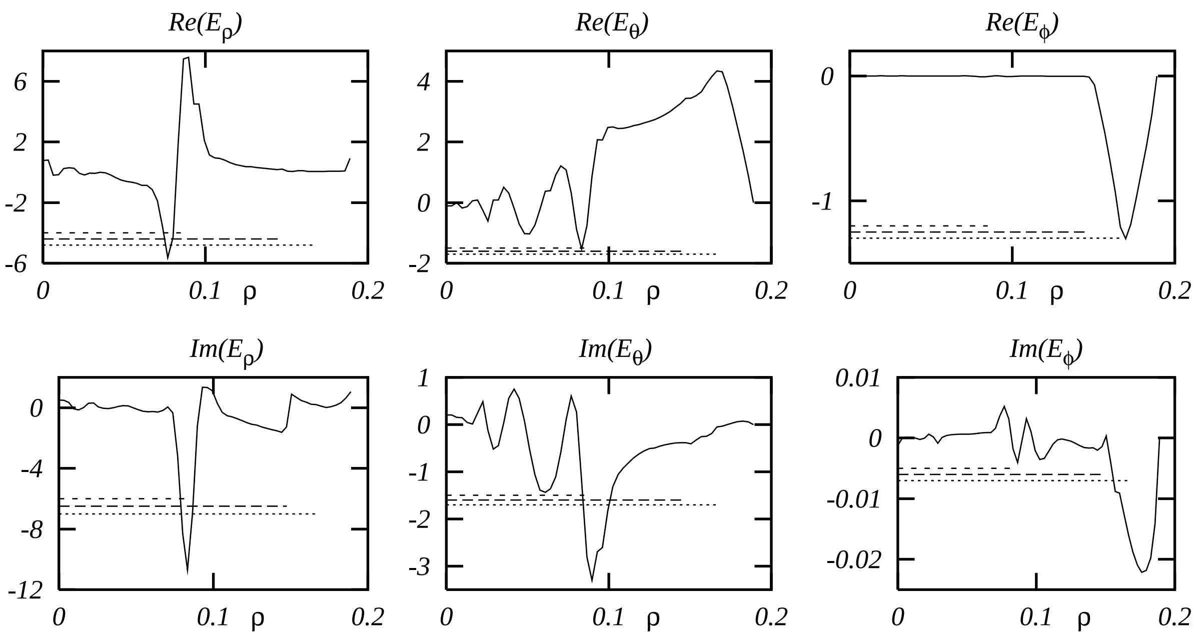

Figure 2. Equatorial profiles of the solutions shown in Fig. 1.