GLOBAL STABILITY OF STELLARATOR CONFIGURATIONS

L. Garcia

Universidad Carlos III. 28911 Leganés, Madrid. SPAIN

The MHD stability of stellarator configurations is examined using a set of resistive MHD equations derived for 3D general configurations. Up to now, the calculation of global modes has been based on formulations of the ideal MHD energy principle in magnetic coordinates [1, 2]. Here, we develop a formalism to solve the full MHD equations for either ideal or resistive modes.

We begin with the usual compressible MHD equations, namely

$$\frac{\partial \mathbf{A}}{\partial t} = -\nabla \alpha + \mathbf{v} \times \mathbf{B} - \eta \mathbf{J}, \tag{1}$$

$$\rho_m \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{J} \times \mathbf{B}, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \Gamma p \nabla \cdot \mathbf{v} = 0, \tag{3}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{4}$$

$$\mathbf{v} = \nabla \times \mathbf{\Omega} + \nabla \omega, \tag{5}$$

where the magnetic field and the velocity are written in terms of potentials, α is the electrostatic potential, Γ is the specific heats ratio, and the mass density ρ_m is assumed to be constant.

The MHD equations are solved in toroidal geometry, and we use magnetic Boozer coordinates [3] based on the equilibrium as coordinate system. The gauge for the potentials is chosen to be

$$A_\rho = \Omega_\rho = 0. \tag{6}$$

We study first the equations in the pressure convection limit ($\Gamma = 0$) with the assumption $\omega = 0$. By taking the curl of the linear momentum balance equation multiplied by the jacobian, we obtain:

$$\frac{\partial \mathbf{U}}{\partial t} = -\sqrt{g} \nabla \sqrt{g} \times \nabla p + \sqrt{g} \nabla \times \left(\sqrt{g} \mathbf{J} \times \mathbf{B} \right), \tag{7}$$

where $\mathbf{U} = \rho_m \sqrt{g} \nabla \times \left(\sqrt{g} \mathbf{v} \right)$.

With the preceding specifications and the definitions $A_\zeta = -\psi$ (the poloidal magnetic flux function), $A_\theta = -\chi$ (the toroidal magnetic flux function), $\Omega_\zeta = -\Phi$ (the poloidal stream function), and $\Omega_\theta = -\Lambda$ (the toroidal stream function), the equations that are solved are (in dimensionless form):

$$\frac{\partial \psi}{\partial t}=\frac{\partial \alpha}{\partial \zeta}-\frac{\rho \iota}{\sqrt{g}}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}-\frac{\partial \Lambda}{\partial \zeta}\right)+\eta J_{\zeta}. \tag{8}$$

$$\frac{\partial \chi}{\partial t}=\frac{1}{\rho}\frac{\partial \alpha}{\partial \theta}-\frac{1}{\sqrt{g}}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}-\frac{\partial \Lambda}{\partial \zeta}\right)+\eta J_{\theta}, \tag{9}$$

$$0=-\frac{\partial \alpha}{\partial \rho}+\frac{1}{\sqrt{g}}\left[\frac{\partial \Phi}{\partial \rho}-\iota\frac{\partial(\rho\Lambda)}{\partial \rho}\right]-\eta J_{\rho}, \tag{10}$$

$$\begin{aligned}\frac{\partial U^{\theta}}{\partial t}&=S^2\frac{\beta_0}{2\varepsilon^2}\left(\frac{\partial \sqrt{g}}{\partial \rho}\frac{\partial p}{\partial \zeta}-\frac{\partial \sqrt{g}}{\partial \zeta}\frac{\partial p}{\partial \rho}\right)\\&\quad+S^2\left\{\frac{\partial}{\partial \zeta}\left[g\left(J^{\theta}B^{\zeta}-J^{\zeta}B^{\theta}\right)\right]-\frac{\partial}{\partial \rho}\left[g\left(J^{\rho}B^{\theta}-J^{\theta}B^{\rho}\right)\right]\right\},\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial U^{\zeta}}{\partial t}&=S^2\frac{\beta_0}{2\varepsilon^2}\left(\frac{1}{\rho}\frac{\partial \sqrt{g}}{\partial \theta}\frac{\partial p}{\partial \rho}-\frac{\partial \sqrt{g}}{\partial \rho}\frac{1}{\rho}\frac{\partial p}{\partial \theta}\right)\\&\quad+S^2\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho g\left(J^{\zeta}B^{\rho}-J^{\rho}B^{\zeta}\right)\right]-\frac{1}{\rho}\frac{\partial}{\partial \theta}\left[g\left(J^{\theta}B^{\zeta}-J^{\zeta}B^{\theta}\right)\right]\right\},\end{aligned} \tag{12}$$

$$\frac{\partial p}{\partial t}=\frac{dp_{eq}}{d\rho}\frac{1}{\sqrt{g}}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}-\frac{\partial \Lambda}{\partial \zeta}\right), \tag{13}$$

where

$$J^{\rho}=\frac{1}{\sqrt{g}}\left(\frac{1}{\rho}\frac{\partial B_{\zeta}}{\partial \theta}-\frac{\partial B_{\theta}}{\partial \zeta}\right),\ J^{\theta}=\frac{1}{\sqrt{g}}\left(\frac{\partial B_{\rho}}{\partial \zeta}-\frac{\partial B_{\zeta}}{\partial \rho}\right),\ J^{\zeta}=\frac{1}{\sqrt{g}}\left[\frac{1}{\rho}\frac{\partial(\rho B_{\theta})}{\partial \rho}-\frac{1}{\rho}\frac{\partial B_{\rho}}{\partial \theta}\right], \tag{14}$$

$$B^{\rho}=\frac{1}{\sqrt{g}}\left(\frac{\partial \chi}{\partial \zeta}-\frac{1}{\rho}\frac{\partial \psi}{\partial \theta}\right),\ B^{\theta}=\frac{1}{\sqrt{g}}\frac{\partial \psi}{\partial \rho},\ B^{\zeta}=-\frac{1}{\sqrt{g}}\frac{1}{\rho}\frac{\partial(\rho\chi)}{\partial \rho}, \tag{15}$$

$$U^{\theta}=\frac{\partial}{\partial \zeta}\left(\sqrt{g}v_{\rho}\right)-\frac{\partial}{\partial \rho}\left(\sqrt{g}v_{\zeta}\right),\ U^{\zeta}=\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\sqrt{g}v_{\theta}\right)-\frac{1}{\rho}\frac{\partial}{\partial \theta}\left(\sqrt{g}v_{\rho}\right), \tag{16}$$

$$v^{\rho}=\frac{1}{\sqrt{g}}\left(\frac{\partial \Lambda}{\partial \zeta}-\frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}\right),\ v^{\theta}=\frac{1}{\sqrt{g}}\frac{\partial \Phi}{\partial \rho},\ v^{\zeta}=-\frac{1}{\sqrt{g}}\frac{1}{\rho}\frac{\partial(\rho\Lambda)}{\partial \rho}. \tag{17}$$

In Eqs. (8)-(17), all lengths are normalized to the minor radius a, the resisitivity and the pressure to their equilibrium value at the magnetic axis, the magnetic field to the vacuum field at the magnetic axis, and the time to the resistive diffusion time $\tau_R = \mu_0 a^2/\eta$, $S = \tau_R/\tau_{hp}$ is the Lundquist number, and $\tau_{hp} = R_0(\mu_0\rho_m)^{1/2}/B_0$ is the poloidal Alfvén time.

We assume a perfect conducting wall boundary condition at the plasma edge ($\rho = 1$). This implies the following boundary conditions:

$$B^{\rho}(1)=v^{\rho}(1)=\alpha(1)=p(1)=0. \tag{18}$$

To solve Eqs. (8)-(13), the perturbed quantities are expanded in Fourier series in the generalized poloidal and toroidal angles. The equations are time-advanced using the numerical method described in Refs. [4, 5] for tokamaks. The problem is now more complex due to the coupling of different toroidal mode numbers since the equilibrium contains not only $n = 0$ but all the multiples of the number of field periods.

We also consider the incompressible limit. By taking the curl of the linear momentum balance equation, we obtain:

$$\frac{\partial \mathbf{U}}{\partial t} = \sqrt{g}\boldsymbol{\nabla}\times(\mathbf{J}\times\mathbf{B}), \tag{19}$$

where $\mathbf{U} = \rho_m \sqrt{g}\boldsymbol{\nabla}\times\mathbf{v}$. Eqs. (11), (12) are changed accordingly, and Eq. (13) is no longer needed since the pressure does not appear in the system of equations.

We have studied the ideal stability of an LHD equilibrium with a 15-cm inward shift. Details of this magnetic configuration can be found in Ref. [6]. The equilibrium pressure distribution is $p = p_0\left(1-\psi^2\right)^2$, with ψ a normalized poloidal flux. With this profile, relatively strong instabilities which are localized near ι = 2/3 can appear. Since LHD has 10 field periods, there are six different mode families generated by the beating of equilibrium and perturbation [2]. Here, we analyze the stability of the n = 2, 8, 12, 18, ... family. Good convergence is obtained by including n = 0, 10, 20 equilibrium components in the calculation. When the number of field periods is high, each (m, n) component resonant in the plasma is weakly coupled to Fourier components of the same family also resonant, but with different n-value.

Fig. 1 shows the mode structure for the n = 2 mode. In this case, the resonant n = 2 components are included in the calculation, but the resonant $n \neq 2$ components are not. Fig. 2 is the same as Fig. 1 for the n = 38 mode.

The growth rate of the n = 2 mode vs. β_0 is shown in Fig. 3 for both the pressure convection limit and the incompressible limit. The results are compared with those obtained using the stellarator expansion [6]. The dependence of the growth rate with n is shown in Fig. 4. The difference between the full equations and the stellarator expansion is due to the coupling with other n-values in the family. When only the n = 0 equilibrium components are included in the calculation, the results are very similar.

Acknowledgments

We are very grateful to S.P. Hirshman for providing us with the VMEC code. This work was done under financial support from DGES Project No. PB96-0112-C02-01.

References

[1] D.V. Anderson, et al, in *Proceedings, Joint Varenna-Lausanne International Workshop on Theory of Fusion Plasmas* (Compositori, Bologna, 1988), p. 93.

[2] C. Schwab, Phys. Fluids B **5**, 3195 (1993).
[3] A.H. Boozer, Phys. Fluids **23**, 904 (1980).
[4] L.A. Charlton, et al, J. Comp. Phys. **86**, 270 (1990).
[5] L.A. Charlton, et al, J. Comp. Phys. **63**, 107 (1986).
[6] Y. Nakamura, et al., J. Comp. Phys. **128**, 43 (1996).

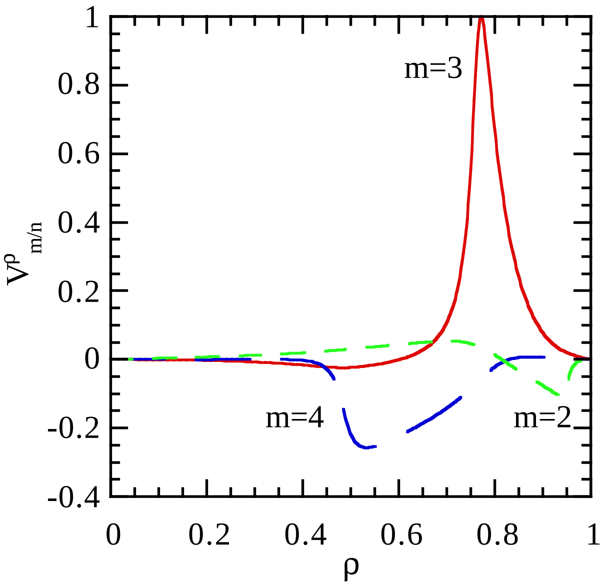

Fig. 1. Dominant poloidal components of the n = 2 mode for the equilibrium with β_0 = 4% (pressure convection limit).

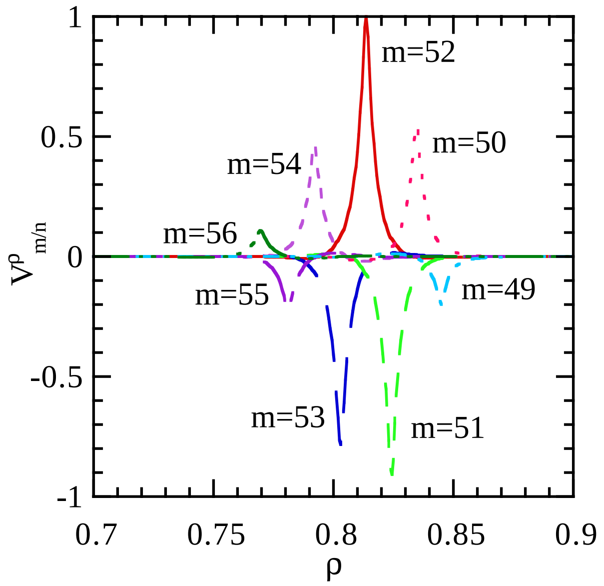

Fig. 2. The same as Fig. 1 for the n = 38 mode.

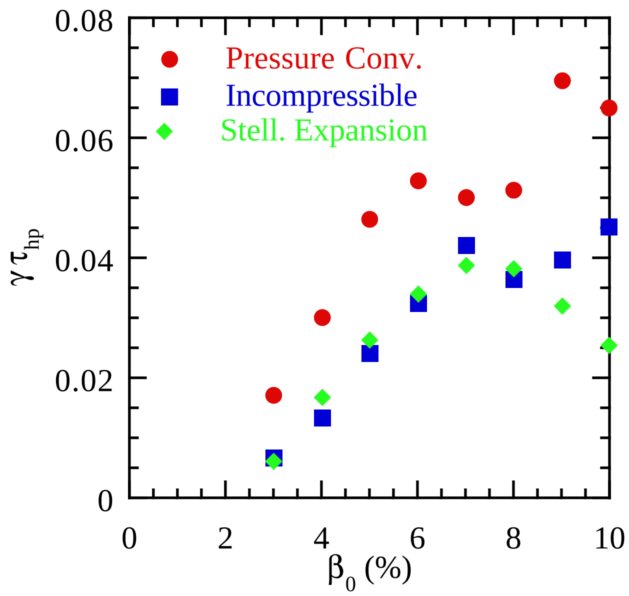

Fig. 3. Growth rate vs. β for the different models.

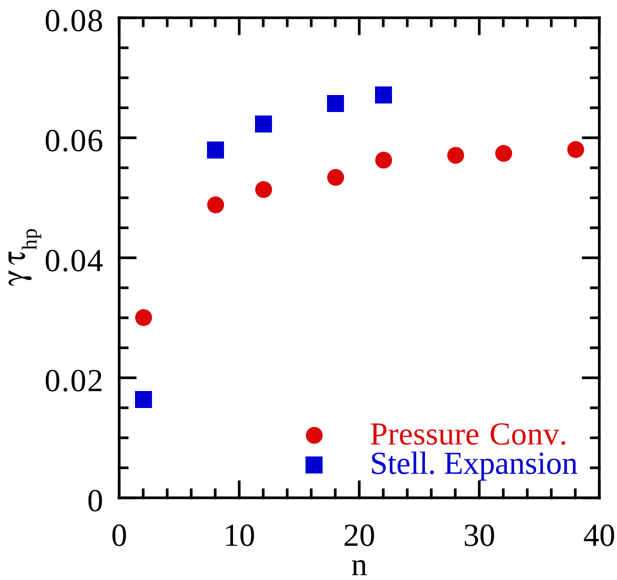

Fig. 4. Growth rate vs n for the pressure convection limit and the stellarator expansion (β_0 = 4%).