

TOKAMAK TRANSPORT FOR ARBITRARY MAGNETIC CROSS SECTION IN THE COLLISIONAL REGIME. APPLICATIONS TO SOME PARTICULAR CASES

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Abstract

The Pfirsch-Schlüter transport is analyzed here for cross sections of arbitrary form. The analysis uses a system of coordinates described previously. Inductive electric field E_φ is included in our analysis, and toroidal axisymmetry is assumed. The average normal velocity is obtained as function of several closed integrals around the intersection curve of a magnetic surface with a meridian plane. The integrals have been calculated for the particular geometries of elliptic cross sections. Agreement with previous results has been found for the large aspect ratio case, and for previous analysis with elliptic geometries. The contribution of the terms due to E_φ are analyzed in detail.

1. Introduction

In the collisional regime fluid equations give an appropriate description for transport in tokamaks. Thus the treatment using resistive MHD equations can be applied and the results are reliable. Pfirsch and Schlüter [1] first and several other authors later analyzed [2-5], this regime in the large aspect ratio approximation. On the other hand Soloviev [6] and later P. Martin [7] discussed transport in this regime for elliptic cross sections. However a general analysis for arbitrary cross sections including toroidal electric field E_φ has not been performed previously. In the present paper we describe the equations for general cross sections assuming toroidal symmetry and neglecting radial temperature gradient as well as the non-linear terms $\vec{v} \cdot \nabla \vec{v}$ in the momentum equation. Future analysis including these effects will be published elsewhere, thus in this paper the confinement effects due to plasma rotation will not be included. Our general analysis will be applied to some particular cases as circling cross sections in the large aspect approximation and elliptic cross sections. In the first case the contribution of E_φ cancel each other and in the case of elliptic section in the limit of zero toroidal electric field, previous results are obtained [7]. Our treatment follows the line of previous papers [7-9], where families of magnetic field lines and families of orthogonal curves were used to describe the topology of the magnetic surfaces.

2. Theoretical Treatment

We are interested in the average normal velocity \bar{v} across the magnetic surfaces. Generalizing Eq. (4.3.1) of Ref. [7], this velocity \bar{v} is given by

$$\bar{v} = \frac{1}{S_T} \oint \left(\frac{\vec{E} \times \vec{B}}{B^2} - \eta_{\perp} \frac{\nabla p}{B^2} \right) \cdot (-\hat{n}) dS \quad (1)$$

where \hat{n} is the normal to a determined magnetic surface (see Fig 1 of Ref.[8]) and S_T is the area of this total magnetic surface. Now introducing the components of \vec{E} in the Frenet frame $(\hat{t}, \hat{n}, \hat{\phi})$ of the curve intersection of the magnetic field surfaces with a meridian plane.

$$\vec{E} = E_{\sigma} \hat{t}_{\sigma} + E_p \hat{t} + E_{\varphi} \hat{\phi} = -E_{\sigma} \hat{n} + E_p \hat{t} + E_{\varphi} \hat{\phi}$$

$$\bar{v} = -\frac{1}{\oint R ds} \oint \left(\eta_{\perp} \frac{R}{B^2} \frac{\partial p}{\partial \sigma} - \frac{R E_p B_{\varphi}}{B^2} + \frac{R E_{\varphi} B_p}{B^2} \right) ds \quad (2)$$

Here as in previous papers [7-9] we are denoting with s and σ , respectively the arcs along the above magnetic field intersection curve and the family of orthogonal field curves. From the equations

$$\nabla \cdot \vec{J} = \nabla \cdot \left(j_{\parallel} \frac{\vec{B}}{B} + \vec{J}_{\perp} \right) = 0 \quad (3)$$

and

$$\vec{J} \times \vec{B} = \nabla p \quad (4)$$

we get

$$\frac{\partial}{\partial s} \left(\frac{j_{\parallel}}{B} \right) = -\frac{B_{\varphi}}{B_p} \frac{\partial p}{\partial \sigma} \frac{\partial}{\partial s} \left(\frac{1}{B^2} \right) \quad (5)$$

Now after a long algebraic manipulation of the MHD equations the following relation can be proved

$$\frac{\partial}{\partial s} \left(\frac{B_{\varphi}}{B_p} \frac{\partial p}{\partial \sigma} \right) = 0 \quad (6)$$

In this way it is obtained

$$j_{\parallel} = -\frac{B_{\varphi}}{B_p B} \frac{\partial p}{\partial \sigma} + F(\sigma) B \quad (7)$$

Besides, the electric field \vec{E} is produced by the electric potential as well as the inductive electric field E_{φ} . Thus

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} = -\nabla \phi - \frac{\partial A_{\varphi}}{\partial t} \hat{\phi} = -\nabla \phi + E_{\varphi} \hat{\phi} \quad (8)$$

Considering now the parallel part of the Ohm's Law

$$\vec{E} \cdot \vec{B} = E_{\varphi} B_{\varphi} - \vec{B} \cdot \nabla \phi = \eta_{\parallel} j_{\parallel} B \quad (9)$$

Since ϕ must be univalued

$$\oint \frac{\partial \phi}{\partial s} ds = -\oint \eta \frac{j_{\parallel} B}{B_p} ds + \oint \frac{E_{\varphi} B_{\varphi}}{B_p} ds = 0 \quad (10)$$

In this way $F(\sigma)$ is determined and later j_{\parallel} and E_p . The results are not easy to write in a short way, but introducing everything in the equation for \bar{v} , it is obtained

$$v = -\frac{1}{I_0} \left(\frac{\partial p}{\partial \sigma} \right)_1 \eta_{\perp} \frac{R_1}{B_{p1}} \left[\frac{I_2}{R_1^2} + \frac{\eta_{\parallel}}{\eta_{\perp}} B_{\varphi 1}^2 \left(I_3 - \frac{I_1^2}{I_4} \right) \right] - \frac{R_1 E_{\varphi 1}}{I_0} \left[I_5 - B_{\varphi 1}^2 R_1^2 \left(\frac{I_1 I_6}{I_4} - I_7 \right) \right] \quad (11)$$

where $I_i (i = 0, 1, 2, 3, 4, 5, 7)$ are closed line integrals defined as:

$$I_0 = \oint R ds; \quad I_1 = \oint \frac{1}{B_p} ds; \quad I_2 = \oint \frac{R^2 B_p}{B^2} ds; \quad I_3 = \oint \frac{1}{B_p B^2} ds; \quad I_4 = \oint \frac{B^2}{B_p} ds$$

$$I_5 = \oint \frac{B_p}{B^2} ds; \quad I_6 = \oint \frac{ds}{R^2 B_p}; \quad I_7 = \oint \frac{ds}{R^2 B_p B^2} \quad (12)$$

and the integrals are around the curve obtained as the intersection of a meridian plane with a magnetic surface. In the equation (10) the quantities $\left(\frac{\partial p}{\partial \sigma} \right)_1$, R_1 , B_{p1} , $E_{\varphi 1}$, and $B_{\varphi 1}$ must be evaluated at the same point P_1 which is usually the point for $\theta = 0$. However any point in the magnetic surface is good, because

$$\frac{\partial}{\partial s} (B_{\varphi} R) = 0; \quad \frac{\partial}{\partial s} (E_{\varphi} R) = 0; \quad \frac{\partial}{\partial s} \left[\frac{\partial p}{\partial \sigma} \frac{R}{B_p} \right] = 0.$$

with the approximations here used.

In the case of a family of nested elliptic magnetic cross-sections

$$R = R_0 + \lambda b \cos \theta = R_0 (1 + b_* \cos \theta)$$

$$z = \lambda a \sin \theta = R_0 a_* \sin \theta$$

the previous integrals can be performed. In this case the integrals from I_0 to I_4 were obtained previously in Ref. [7], the new integrals are now

$$I_5 = 2\pi \frac{R_0^2 a_* \delta}{R_1 B_{\varphi 1}} \left[1 - \frac{\delta^2}{\sqrt{(1 + \delta^2)(\gamma^2 + \delta^2)}} \right]; \quad I_6 = \frac{2\pi a_*}{B_{\varphi 1} R_1 b_*^2 \delta} \left(\frac{1}{\sqrt{1 - b_*^2}} - 1 \right) (1 + \gamma^2 \sqrt{1 - b_*^2})$$

$$I_7 = \frac{2\pi R_0^2 a_*}{R_1^3 B_{\varphi 1}^3 \delta} \left[\frac{1 + \gamma^2}{2} - \delta^2 + \frac{\delta^4}{\sqrt{(1 + \delta^2)(\gamma^2 + \delta^2)}} \right]$$

In the large aspect ratio approximation, that is, when the ellipses become circles ($b_* = a^*$ and $\gamma = 1$), the integrals I_5 , I_6 and I_7 become

$$I_5 \simeq \frac{2\pi R_0^2 a_* \delta}{R_1 B_{\varphi 1} (1 + \delta^2)}; \quad I_6 \simeq \frac{2\pi a_*}{B_{\varphi 1} R_1 \delta \sqrt{1 - a_*^2}}; \quad I_7 \simeq \frac{2\pi R_0^2 a_*}{R_1^3 B_{\varphi 1}^3 \delta (1 + \delta^2)}$$

then we note that the contribution of each term is of the form $\frac{E_{\varphi 1}}{E_{p1}}$ as in the Ware pinch [10-12], however the contributions of the three terms with $E_{\varphi 1}$ cancelled each other, thus giving a zero

the contribution. In the present treatment the main term of the guiding center is $\frac{\vec{E} \times \vec{B}}{B^2}$, and this does not produce any diffusion. In order to obtain the Ware pinch, the gradient drift should be included as it was proved recently [12].

It is interesting to point out that in the general case of elliptic cross section, the terms with E_φ also contributes to transport, however their contribution seems less important than the Ware pinch effect.

Considering all terms together the velocity diffusion in the large aspect ratio approximation are in agreement with that of previous authors [1-6].

3. Conclusion

Here the velocity diffusion in the collisional regime is obtained for arbitrary magnetic field cross sections using resistive MHD equations and neglecting radial temperature gradient as well as the non-linear terms $\vec{v} \cdot \nabla \vec{v}$. Toroidal axisymmetry is assumed. The contribution due to E_φ is due to several sources one coming from the additional contribution to v , $-E_\varphi B_p / B^2$, and the others from the Ohm's Law equation. These contributions in general have different signs and thus can oppose each other. The application to some particular case as elliptic cross sections shows this feature. Analytic results can be obtained for the case family of nested ellipses and the calculation has been carried out in detail here. In the case of zero toroidal electric field our results coincide with those of previous works [7]. Looking the large aspect ratio approximation the contribution of each term coming from E_φ reminds the typical Ware pinch form $\frac{E_\varphi}{s_p}$, however one term cancelled the others and the total contribution of this part is zero, therefore the classical Pfirsch-Schlüter result [1, 7] is obtained.

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