

# TOROIDAL MAGNETOHYDRODYNAMIC EQUILIBRIA WITH INCOMPRESSIBLE FLOWS

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## 1. Introduction and conclusions

In a recent work [2] we found that the equilibrium differential equation of a cylindrical plasma with incompressible flows and arbitrary cross sectional shape is amenable to a variety of analytic solutions. Also, in the physically appealing case of plasmas with isothermal magnetic surfaces their cross sections are restricted to be circular in the large. The present paper aims at extending the study to the most interesting case of axisymmetric plasmas. It turns out that, as the case in cylindrical plasmas, the incompressibility condition results in a considerable simplification of the problem, i.e., the equilibrium equations reduce to an elliptic partial differential equation along with a relation for the pressure. When the modulus of the Mach number of the poloidal velocity with respect to the poloidal-magnetic-field Alfvén velocity [ $M^2 \equiv v_p^2/(B_p^2/\rho)$ ] takes constant values, several classes of exact equilibria can be derived including those with non-vanishing poloidal and toroidal flows and differentially varying radial electric fields. This is the subject of Sec. 2. Incompressible  $T = T(\psi)$  equilibria are then examined in Sec. 3. Unlike the case in cylindrical plasmas, no restriction appears on the shapes of the magnetic surfaces in axisymmetric equilibria with incompressible flows and isothermal magnetic surfaces, except that they have to be circular locally near the magnetic axis. The latter equilibria satisfy a set of six ordinary differential equations which for flows parallel to the magnetic field  $\mathbf{B}$  can be solved senianalytically. In addition, it is proved the non existence of incompressible axisymmetric equilibria with (a) purely poloidal flows and (b) non-parallel flows with isothermal magnetic surfaces and  $|\mathbf{B}| = |\mathbf{B}|(\psi)$  (omnigenous equilibria).

## 2. Equilibrium equations and exact solutions.

The ideal magnetohydrodynamic (MHD) equilibrium states of plasma flows are governed by the equations  $\nabla \cdot (\rho \mathbf{v}) = 0$ ,  $\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla P$ ,  $\nabla \times \mathbf{E} = 0$ ,  $\nabla \times \mathbf{B} = \mathbf{j}$ ,  $\nabla \cdot \mathbf{B} = 0$ , and  $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ , written in standard notations and convenient units. For an axially symmetric magnetically confined plasma with flow the divergence-free fields can be expressed in terms of the stream functions  $\psi(R, z)$ ,  $I(R, z)$ ,  $F(R, z)$  and  $\Theta(R, z)$  as  $\mathbf{B} = I\nabla\phi + \nabla\phi \times \nabla\psi$ ,  $\mathbf{j} = \Delta^*\psi\nabla\phi - \nabla\phi \times \nabla I$  and  $\rho\mathbf{v} = \Theta\nabla\phi + \nabla\phi \times \nabla F$ . Here,  $R, \phi, z$  are cylindrical coordinates with  $z$  corresponding to the axis of symmetry, constant  $\psi$  surfaces are the magnetic surfaces and  $\Delta^*$  is the elliptic operator defined by  $\Delta^* = R^2\nabla \cdot (\nabla/R^2)$ . The above set of starting MHD equations can be reduced by means of certain integrals of the system, which are shown to be

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surface quantities. To identify three of these quantities, the time independent electric field is expressed by  $\mathbf{E} = -\nabla\Phi$  and the Ohm's law is projected along  $\nabla\phi$ ,  $\mathbf{B}$  and  $|\nabla\psi|$ , respectively, yielding  $F = F(\psi)$ ,  $\Phi = \Phi(\psi)$  and

$$\frac{1}{\rho R^2}(IF' - \Theta) = \Phi'. \quad (1)$$

(The prime denotes differentiation with respect to  $\psi$ ). A fourth surface quantity is derived from the component of equation  $\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla P$ , along  $\nabla\phi$ :

$$I \left( 1 - \frac{(F')^2}{\rho} \right) + R^2 F' \Phi' \equiv X(\psi). \quad (2)$$

From Eq. (2) it follows that, unlike the case in static equilibria,  $I$  is not a surface quantity. With the aid of the above surface quantities the components of the momentum conservation equation along  $\mathbf{B}$  and perpendicular to a magnetic surface are put in the respective forms

$$\mathbf{B} \cdot \left[ \nabla \left( \frac{v^2}{2} + \frac{\Theta}{\rho} \Phi' \right) + \frac{\nabla P}{\rho} \right] = 0 \quad (3)$$

and

$$\begin{aligned} & \left\{ \nabla \cdot \left[ \left( 1 - \frac{(F')^2}{\rho} \right) \frac{\nabla\psi}{R^2} \right] + \frac{F'' F'}{\rho} \frac{|\nabla\psi|^2}{R^2} \right\} |\nabla\psi|^2 \\ & + \left[ \frac{\rho}{2} \left( \nabla v^2 - \frac{\nabla(\Theta/\rho)^2}{R^2} \right) + \frac{\nabla(I^2)}{2R^2} + \nabla P \right] \cdot \nabla\psi = 0 \end{aligned} \quad (4)$$

It is pointed out that Eqs. (3) and (4) hold for any equation of state.

In order to reduce the equilibrium equations further, we employ the incompressibility condition  $\nabla \cdot \mathbf{v} = 0$ , which on account of  $\nabla \cdot (\rho\mathbf{v}) = 0$  implies that the density is a surface quantity. Consequently, from Eqs. (1) and (2) it follows that, unlike the case in cylindrical plasmas [1], axisymmetric incompressible equilibria with purely poloidal flows ( $\Theta = 0$ ) can not exist; the only possible equilibria of this kind are of cylindrical shape. With the aid of  $\rho = \rho(\psi)$ , Eq. (3) can be integrated to yield an expression for the pressure, i.e.

$$P = P_s(\psi) - \rho \left( \frac{v^2}{2} + \frac{\Phi'\Theta}{\rho} \right). \quad (5)$$

We note here that, unlike in static equilibria [ $P = P_s(\psi)$ ], in the presence of flow magnetic surfaces in general do not coincide with isobaric surfaces, as also expected from the momentum conservation equation. If it is now assumed that  $M^2 = (F')^2/\rho \neq 1$  and Eq. (5) is inserted into Eq. (4), the latter reduces to the *elliptic* differential equation

$$\begin{aligned} & (1 - M^2) \Delta^* \psi - \frac{1}{2} (M^2)' |\nabla\psi|^2 \\ & + \frac{1}{2} \left( \frac{X^2}{1 - M^2} \right)' + R^2 \left( P_s - \frac{X F' \Phi'}{1 - M^2} \right)' + \frac{R^4}{2} \left( \frac{\rho (\Phi')^2}{1 - M^2} \right)' = 0. \end{aligned} \quad (6)$$

Equation (6), containing the surface quantities  $F(\psi)$ ,  $\Phi(\psi)$ ,  $X(\psi)$ ,  $\rho(\psi)$  and  $P_s(\psi)$ , in conjunction with Eq. (5) for the pressure and appropriate boundary conditions determines the equilibrium states of an axisymmetric plasma with incompressible flows.

With the ansatz  $(F')^2/\rho \equiv M_c^2 = \text{const.}$  Eq. (6) reduces to

$$\Delta^* \psi + \frac{1}{(1 - M_c^2)^2} \left[ XX' + R^2 \left( (1 - M_c^2) P_s - XF' \Phi' \right)' + \frac{R^4}{2} (\rho (\Phi')^2)' \right] = 0. \quad (7)$$

Eq. (7) can be linearized and solved for (a) purely toroidal flows ( $M_c^2 = F' = 0$ ) (b) flows parallel to  $\mathbf{B}$  ( $\Phi' = 0$ ) and non-parallel flows. The latter equilibria are of particular interest because non-parallel flows with non-vanishing poloidal components are associated with radial electric fields which play a role in the transitions to improved confinement regimes. As an example we derive here a class of equilibria of this kind by the ansatz  $\Phi' \propto \psi^{-k/2}$  and  $\rho \propto \psi^k$ , where  $k$  is a parameter. The electric field is then of the form  $\mathbf{E} \propto -\psi^{-k/2} \nabla \psi$  and Eq. (7) becomes  $\Delta^* \psi + \hat{X} \hat{X}' + R^2 [\hat{P}_s - d_0 \hat{X}]' = 0$  where  $d_0 = \text{const.}$  The simplest solution of this equation ( $\hat{X} = \hat{X}_0 = \text{const.}$ ) is  $\psi = \psi_c (R^2/R_c^4) (2R_c^2 - R^2 - 4d^2 z^2)$ , where  $\psi_c$  is the value of the flux function at the position of the magnetic axis ( $z = 0$ ,  $R = R_c$ ) and  $d^2$  is a parameter related to the shape of the flux surfaces. It is noted that according to Eq. (2), the toroidal magnetic field, in addition to the usual  $1/R$  component, contains a flow term linear in  $R$ :  $B_\phi = I/R = \hat{X}_0/R - d_0 R$ . Thus, the modification of  $B_\phi$  may affect the shape of the safety factor profile. This indicates that the flow along with the associated radial electric field may contribute to the creation of improved confinement regimes related to appropriate shaping of the safety factor profiles, e.g. inverse-magnetic-shear profiles.

### 3. $T = T(\psi)$ equilibria.

For fusion plasmas the thermal conduction along  $\mathbf{B}$  is fast compared to the heat transport perpendicular to a magnetic surface and therefore equilibria with isothermal magnetic surfaces are of particular interest. It is noted that for cylindrical plasmas the relation  $T = T(\psi)$  restricts the cross sections of the magnetic surfaces to be circular [2]. We shall show that axisymmetric incompressible  $T = T(\psi)$  equilibria are free of such a restriction except near the magnetic axis.

Under the assumption that the plasma obeys to the ideal gas law  $P = \hat{R} \rho T$ , Eq. (5) leads to the following expression for the magnetic field modulus:

$$|\mathbf{B}|^2 = \Xi(\psi) + R^2 H(\psi), \quad (8)$$

where  $\Xi(\psi) \equiv 2(P_s - P)\rho/(F')^2$  and  $H(\psi) \equiv (\rho \Phi'/F')^2$ . Consequently, apart from the case of field aligned flows ( $H = 0$ ), omnigenous equilibria, viz. equilibria with  $|\mathbf{B}|$  being a surface quantity, are not possible. Solving the set of equations (4) and (8) for  $|\nabla \psi|^2$  and  $\Delta^* \psi$  one obtains

$$\left( \frac{\partial \psi}{\partial R} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 = 2(i(\psi) + R^2 j(\psi) + R^4 k(\psi)) \quad (9)$$

and

$$\frac{\partial^2 \psi}{\partial^2 R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial^2 z^2} = -f(\psi) - R^2 g(\psi) - R^4 h(\psi), \quad (10)$$

where  $i(\psi)$ ,  $j(\psi)$ ,  $k(\psi)$ ,  $f(\psi)$ ,  $g(\psi)$  and  $h(\psi)$  are known functions of  $\psi$ . With the introduction of the quantities  $x \equiv R^2$ ,  $p = \partial\psi/\partial x$ ,  $q = \partial\psi/\partial z$ ,  $r = \partial^2\psi/\partial x^2$  and  $t = \partial^2\psi/\partial z^2$  Eqs. (9) and (10) are written in the respective forms

$$4xp^2 + q^2 = 2(i + xj + x^2k) \quad (11)$$

and

$$4xr + t = -f - xg - x^2h. \quad (12)$$

To integrate Eqs. (11) and (12) we apply a procedure suggested by Palumbo [3]. Accordingly, employing  $R$  and  $\psi$  as independent coordinates instead of  $R$  and  $z$  (then  $z = z(x, \psi)$ ), the relation  $f + i' = 0$  (which is satisfied identically) and Eq. (11), Eq. (12) reduces to  $4 \partial p/\partial x|_\psi = -x(g + j') - x^2(h + k')$  and consequently

$$p = -\frac{1}{4}(g + j')x - \frac{1}{8}(h + k')x^2 + \frac{d(\psi)}{4}. \quad (13)$$

Since  $z$  is a function of  $x$  and  $\psi$ , solutions of equation  $dz = -(p/q)dx + (1/q)d\psi$  exist provided  $\partial/\partial\psi(-p/q) = \partial/\partial x(1/q)$ . The latter equation leads to the solvability condition

$$-q^2 \frac{\partial p}{\partial \psi} + \frac{1}{2}p \frac{\partial q^2}{\partial \psi} + \frac{1}{2} \frac{\partial q^2}{\partial x} = 0. \quad (14)$$

Substituting  $q^2$  and  $p$  from Eqs. (11) and (13) in Eq. (14) yields a relation of the form  $\sum_{j=0}^4 a_j(\psi)x^j = 0$ ; hence, since  $x$  and  $\psi$  are independent variables, the equations  $a_j = 0$ , containing the surface functions  $P(\psi)$ ,  $P_s(\psi)$ ,  $F(\psi)$ ,  $\Phi(\psi)$ ,  $X(\psi)$ ,  $\rho(\psi)$  and  $d(\psi)$ , should be satisfied for all  $j$ . If two of these functions are assigned along with boundary conditions the set of Eqs.  $a_j(\psi) = 0$  ( $j = 0, \dots, 4$ ) can be solved numerically. Furthermore, to completely solve the equilibrium problem, one should determine the function  $z(x, \psi)$  which satisfies the equation

$$\left. \frac{\partial z}{\partial x} \right|_\psi = -\frac{p}{q} = \frac{\pm \frac{1}{4} [(g + j')x + \frac{1}{2}(h + k')x^2 - d]}{\left\{ 2(i + xj + x^2k) - \frac{x}{4} [(g + j')x + \frac{1}{2}(h + k')x^2 - d] \right\}^{1/2}}. \quad (15)$$

Consequently,  $z(x, \psi)$  can be expressed in terms of hyper-elliptic integrals. For the particular case of field aligned flows ( $h=k=0$ ) the hyperelliptic integrals reduce to elliptic ones, as suggested in Ref.[4]. At last, on the basis of Eqs. (6) and (8) and an expansion of the surface quantities around the magnetic axis it can be shown that, similar to static omnigenous equilibria [3], the magnetic surfaces near the magnetic axis be circular.

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## References

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