ANALYTICAL STABILITY CONDITION FOR THE IDEAL m = n = 1 KINK MODE IN A TOROIDAL PLASMA WITH ELLIPTIC CROSS SECTION

C. Wahlberg

Dept. of Space and Plasma Physics, Uppsala University, S-755 91 Uppsala, Sweden (EURATOM/NFR Fusion Association)

Introduction Many aspects of the instability responsible for the sawtooth crash are still poorly understood. One such area where the present understanding is incomplete concerns the influence of a non-circular cross section, in particular ellipticity, on the ideal MHD stability of the m = n = 1 kink mode in a toroidal plasma. In the analytical studies by Connor and Hastie [1] and by Bondeson and Bussac [2] a weak, destabilizing effect of the ellipticity was found. The effect was found to be particularly weak when $\Delta q = 1 - q_0$ is small (q_0 is the safety factor at the magnetic axis), with a contribution to the potential energy δW that is quadratic both in the ellipticity and in Δq. This result appears, however, to contradict the numerical results by Lutjens *et al.* [3], who found that the ellipticity has a strongly destabilizing effect, especially when Δq is small. A partial explanation of this effect was given by Lutjens *et al.* in terms of a large aspect ratio expansion of the Mercier criterion, keeping terms in the expansion that include the combined effects of ellipticity, finite pressure and finite aspect ratio. They found a high correlation between Mercier stability and the numerical stability boundaries of the internal kink. It can therefore be expected that terms of this order in the expansion of δW for the ideal m = n =1 mode presumably would have a similar, destabilizing effect. Since terms of this order were not included in the expansions in Refs. [1] and [2], this seems to be the probable reason for the disagreement between these analytical results and the numerical calculations.

In the present work Bussac's *et al.* [4] theory of the internal kink mode in a tokamak is extended to plasmas with an elliptical cross section, keeping terms in the expansion that include the combined effects of ellipticity and toroidicity. This leads to very extensive perturbation expansions, containing of the order of one thousand terms in the highest order. The analysis is therefore to a large extent dependent on the use of computer algebra. The details of the analysis can be found in Ref. [5].

Equilibrium We use a large aspect ratio, low-beta, tokamak equilibrium, expanded up to second order in the inverse aspect ratio ε, i.e. including the natural ellipticity E(r) of the flux surfaces. In addition, we include terms in the expansion that account for the presence of an "external ellipticity" $\widetilde{E}(r)$, produced either by a non-circular wall or by external conductors.

Stability The stability analysis is based on a direct, large aspect ratio expansion of the marginal, linearized, incompressible MHD equations

$$-\nabla\phi + (\mathbf{B}\cdot\nabla)\mathbf{Q} + (\mathbf{Q}\cdot\nabla)\mathbf{B} = 0,\ \ \mathbf{Q} = (\mathbf{B}\cdot\nabla)\xi - (\xi\cdot\nabla)\mathbf{B},\ \ \nabla\cdot\xi = 0. \quad (1)$$

Here ϕ is the perturbed, total pressure, $\mathbf{Q}$ is the perturbed magnetic field, $\mathbf{B}$ is the equilibrium magnetic field and ξ is the plasma perturbation. We start with a perturbation of the form $\exp[i(m\theta - n\varphi)]$, with m = n = 1 to lowest order, and expand these equations up to order $\varepsilon^2 e$, where e is the ellipticity. All side-bands that affect the m = 1 motion up to this order (m = – 1, 0, 2 and 3) are are included in the analysis. As a result, we find that the dynamics of the m = n = 1 mode in a toroidal plasma with elliptic cross section can be modelled by a system of seven coupled differential equations, describing the interaction of m = 1 with the side-band modes above [5]. For of a monotonic q-profile, with $q_0 < 1$, the (normalized) potential energy of the m = 1 mode becomes

$$\delta\widehat{W} = \delta\widehat{W}_{Bussac} + \frac{1}{32}\left(7\widetilde{E}_0 - 3\widetilde{E}'_0\right) - \frac{1}{16}\left(6\beta_p + l_i\right)\left(3\widetilde{E}_0 + \widetilde{E}'_0\right) - \frac{\alpha_2^-\alpha_1^+ + \alpha_2^+\alpha_1^-}{4\left(A_2^- - A_2^+\right)} - \left(\widetilde{E}_0 - \widetilde{E}'_0\right)\times$$

$$\left[\frac{16\alpha_1^+\left(B_3^- + 4\right) - \alpha_1^-\alpha_3}{48\left(A_2^- - A_2^+\right)} + \frac{\alpha_4\left(A_3^- + 4\right) + \alpha_5\left(A_3^+ + 4\right)}{48\left(A_3^- - A_3^+\right)} + \frac{\alpha_6\left(A_3^- + 4\right) - 9B_2^-\alpha_1^-\left(A_3^+ + 4\right)}{48\left(A_2^- - A_2^+\right)\left(A_3^- - A_3^+\right)}\right], \quad (2)$$

where $\delta\widehat{W}_{Bussac} = 1/16 - 3\beta_p/4 - l_i/8 - \alpha_1^-\alpha_1^+/[4(A_2^- - A_2^+)]$ represents the (normalized) potential energy of the m = n = 1 mode in a toroidal plasma with circular cross section [4]. The rest of the quantities in the expression for $\delta\widehat{W}$ above are given by $\alpha_1^\pm = (\Lambda + 1/2)A_2^\pm + 3\Lambda - 3/2$, $\alpha_2^\pm = [(3\Lambda + 1/2)\widetilde{E}_0 - (\Lambda - 1/2)\widetilde{E}'_0]A_2^\pm + (5\Lambda - 9/2)\widetilde{E}_0 + (\Lambda + 3/2)\widetilde{E}'_0$, $\alpha_3 = (27\Lambda + 9)A_2^+ + 69\Lambda - 27$, $\alpha_4 = [12(E_0 - E'_0) - 6\Lambda^2 - 10\Lambda - 4]A_3^+ + 48(E_0 - E'_0) - 72\Lambda^2 + 36\Lambda - 8$, $\alpha_5 = [12(E_0 - E'_0) + 21\Lambda^2 + 39\Lambda/2 + 4]A_3^- + 48(E_0 - E'_0) + 48\Lambda^2 + 69\Lambda - 8 + (9\Lambda + 9/2)B_2^-$ and $\alpha_6 = 16\,(B_3^+\alpha_1^- - B_3^-\alpha_1^+) + 9B_2^+\alpha_1^-$. Furthermore, $\Lambda \equiv \beta_p(r_1) + l_i(r_1)/2$, β_p is the poloidal beta-value, l_i is the plasma inductance, r_1 is the radius of the q = 1 surface, and the quantities E_0, E'_0, $\widetilde{E}_0$ and $\widetilde{E}'_0$ are defined as $E_0 = E(r_1)/r_1^3$, $E'_0 = E'(r_1)/r_1^2$, $\widetilde{E}_0 = \widetilde{E}(r_1)/r_1$ and $\widetilde{E}'_0 = \widetilde{E}'(r_1)$, respectively. In addition, the coefficients $A_m^\pm$ and $B_m^\pm$ are related to the derivatives of the m = 2 and m = 3 side-bands at the q = 1 surface, similarly to the quantities b and c defined in the theory of the internal kink in a plasma with circular cross section [4]. See Ref. [5] for the definition of $A_m^\pm$ and $B_m^\pm$.

In Figs. 1a and b we show the stability limit in β_p as a function of r_1, calculated numerically from the equation $\delta\widehat{W} = 0$, for a few values of the elongation κ, and for the current profiles $j(r) = j_0(1 - r^2)^\nu$ where $\nu = 1$ and 2, respectively. In these figures, β_p and κ refer to the poloidal beta-value and elongation, respectively, at the surface $r = r_1$.

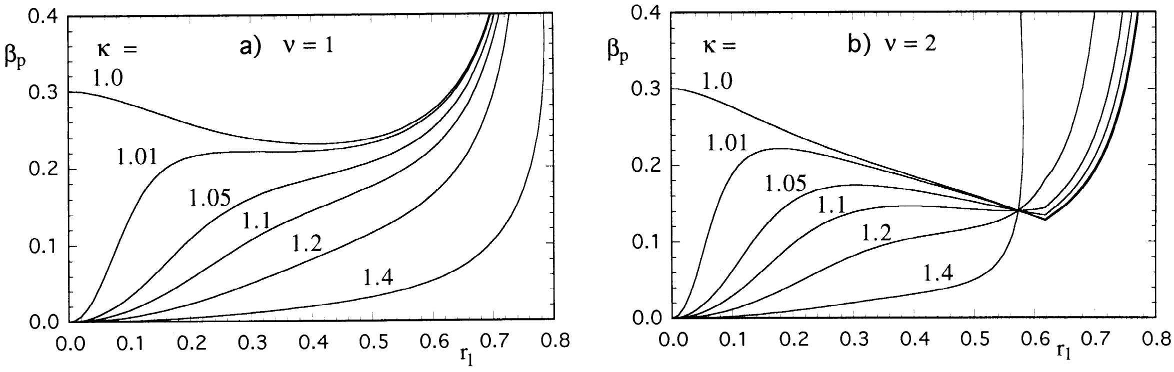

Fig. 1. Critical β_p as a function of the $q = 1$ radius (r_1) for different values of the elongation (κ) and for the current profiles $\nu = 1$ (a) and $\nu = 2$ (b), where $j(r) = j_0(1 - r^2)^\nu$. β_p and κ denote the poloidal beta-value and the elongation, respectively, at the $q = 1$ surface.

The curves with $\kappa = 1$ in the figures represent the results for a plasma with circular cross section, shown also in Fig. 1 in Ref. [4]. It is seen that, for $r_1 \ll 1$, already a very small value of the ellipticity strongly reduces the β_p - limit from the value ≈ 0.3. At larger values of r_1, the stability boundaries for $\kappa - 1 \ll 1$ approach the $\kappa = 1$ curves, while for larger elongations the reduction in β_p is appreciable. In the case $\nu = 2$, however, the ellipticity effect becomes stabilizing when r_1 is larger than a critical value. This has to do with the wall stabilization of the m = 3 side-bands that occur when $q_a < 3$.

The strong, destabilizing effect of the ellipticity when $r_1 \ll 1$ can be understood by expanding the expression for $\delta\widehat{W}$ above in powers of Δq. For a parabolic current profile near the axis one obtains the expansion

$$\delta\widehat{W} = -\frac{3}{4}(\kappa - 1)\beta_p + \Delta q\left[\frac{13}{48} - 3\beta_p^2 + \frac{\kappa - 1}{2}\left(13\beta_p^2 - \frac{1}{4}\beta_p - 1\right)\right] + O(\Delta q^2). \tag{3}$$

The first, Δq-independent term does not depend on the current profile. In the case of a circular cross section, i.e. $\kappa = 1$, the expression above reproduces the result by Bussac *et. al.* [4], leading to the well-known stability condition $\beta_p < (13)^{1/2}/12 \approx 0.3$. In the case of finite ellipticity, however, a significant modification of the result for a circular cross section is obtained. Actually, whenever $\kappa \neq 1$, the first ellipticity term becomes the *dominating* term in δW if Δq is small enough. If $\kappa > 1$, corresponding to vertical elongation of the plasma cross section, the pressure limit becomes zero as $\Delta q \to 0$, whereas for a horizontally elongated plasma ($\kappa < 1$), finite pressure stabilizes the internal kink mode.

Conclusions Bussac's *et al.* [4] theory of the ideal, internal kink mode in a toroidal plasma with circular cross section has been extended to plasmas with an elliptic cross section. It is found that the ellipticity has a strong, destabilizing effect on the internal kink mode. The effect is much larger than the destabilizing ellipticity effect found in the analytical studies by Connor and Hastie [1] and by Bondeson and Bussac [2]. The reason for the difference is the inclusion of the combined ellipticity-toroidicity effect in the present work. In contrast to the ellipticity effect calculated in Refs. [1] and [2], which is insignificant when $\Delta q = 1 - q_0$ is small, the combined ellipticity-toroidicity effect turns out to be particularly important when Δq is small. This is clearly seen in the stability diagrams in Fig. 1, where β_p-critical approaches zero, rather than 0.3, for *any* value of κ greater than unity, as $r_1 \to 0$. This behaviour of β_p-critical can be understood in terms of the Δq-expansion of $\delta\widehat{W}$ in Eq. (3), where the first, Δq-independent, destabilizing ellipticity term dominates $\delta\widehat{W}$ when Δq is small. The values of β_p-critical obtained from the analytical theory are in reasonable agreement with the numerical results found by Lutjens *et al.* [3].

Although the sawtooth collapse is a highly complex phenomenon, where a number of effects not included in the present work are of importance, we think that the ellipticity effect discussed here nevertheless is of such a magnitude that it should be of interest for the interpretation of the sawtooth oscillations in existing tokamaks with an elongated cross section. For instance, the present theory gives a natural explanation for the small values of the poloidal beta value ($\beta_p << 0.3$) observed in JET discharges in connection with the sawtooth collapse in this device [6]. Furthermore, the theory should be of interest in the modelling of the expected sawtooth activity in ITER [7].

Acknowledgment

The author is grateful to Prof. A. Bondeson and Dr. H. G. Eriksson for valuable discussions. This work has been supported by the European Community under an association contract between Euratom and Sweden.

References

[1] J.W. Connor and R.J. Hastie: *The effect of shaped plasma cross sections on the ideal internal kink mode in tokamaks,* Rep. CLM-M-106, Culham Laboratory, (1985).

[2] A. Bondeson and M.-N. Bussac: Nucl. Fusion **32**, 513 (1992).

[3] H. Lütjens, A. Bondeson, and G. Vlad: Nucl. Fusion **32**, 1625 (1992).

[4] M. -N. Bussac, R. Pellat, D. Edery, and J. L. Soulé: Phys. Rev. Lett. **35**, 1638 (1975).

[5] C. Wahlberg: Phys. Plasmas **5**, 1387 (1998).

[6] J. A. Wesson: Plasma Phys. Contr. Fusion **28**, No. 1A, 243 (1986).

[7] F. Porcelli, D. Boucher, and M. N. Rosenbluth, Plasma Phys. Contr. Fusion **38**, 2163 (1996).