RESONANT GUIDING-CENTRE MOTION OF IONS WITH ARBITRARY ORBIT WIDTH IN A TOKAMAK

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1. Introduction

The presence of superthermal ion populations produced by fusion reactions and plasma heating is a characteristic feature of plasmas in modern tokamaks. Ions with large orbit width ($\Delta r_b \sim r$, Δr_b being the radial width of the particle drift orbit, r the radial co-ordinate) constitute significant fractions of such populations. To understand the transport properties of such ions, it is of importance to study the effect of perturbations of the electromagnetic field (waves, instabilities, magnetic field ripple etc.). It is well known that the effect of the perturbations on the resonant particles (i.e., the particles that reach the same perturbation phase after several bounce/transit periods) destroys the drift surfaces, on which the unperturbed trajectories of the particles lie. It leads to appearance of resonance islands near resonant drift surfaces and may cause stochastic diffusion when the perturbation amplitude is sufficiently high for the islands to overlap (we do not cite here the extensive literature on specific examples of resonant behaviour of fast ions).

The aim of this work is to derive the Hamiltonian bounce-averaged equations of the guiding centre motion near a resonant drift surface. An adiabatic invariant of motion near the resonance, which generalizes that found in Ref. [1], is found and used to evaluate the width of the resonance islands and the period of the particle motion around the islands. During the derivation, the action-angle co-ordinates associated with the unperturbed motion [2] are employed, which enables us to separate the bounce motion from the slow motion caused by a perturbation. We use the Lagrangian description of the guiding centre motion in noncanonical co-ordinates [3, 4] for two reasons. First, the use of canonical co-ordinates for description of the guiding-centre motion is, in general, more complicated (although possible, see, e.g., [5]). Second, perturbations of the magnetic field change the symplectic structure of the phase space, i.e., make canonical co-ordinates noncanonical. The non-canonical perturbation techniques [4] are much more flexible and convenient in this situation.

2. The unperturbed particle motion

At first, we study the motion of a particle in the axisymmetric magnetic field of a large-aspect-ratio tokamak, which will be considered here as the unperturbed motion. Neglecting the contribution of the poloidal component to the magnetic field, we present the guiding centre Lagrangian [3] as follows (cf. [1]):

$$\mathcal{L}_u(\vec{x}, \dot{\vec{x}}, \mathcal{E}, t) = \frac{e}{c} F(r)\dot{\vartheta} + J\dot{\varphi} - \mathcal{E}. \qquad (1)$$

where $J = -(e/c)\Psi(r) + Mv_\parallel(r, \vartheta, \mathcal{E})R$ is the canonical angular momentum; $F(r)$ and $\Psi(r)$ are the toroidal and poloidal magnetic fluxes, respectively; $v_\parallel = [2(\mathcal{E} - \mu B_0 R_0/R)/m]^{1/2}$; $\vec{x} = (r, \vartheta, \varphi)$ and $\mathcal{E}$ are the particle location and energy; μ is the magnetic moment treated here as a parameter; R is the distance to the axis of the symmetry; B_0 and R_0 are the magnetic field and R at the magnetic axis; ϑ and φ are the poloidal and toroidal co-ordinates with ϑ chosen so that the magnetic field lines are straight. $\mathcal{E}$ and J are the constants of motion that determine the invariant tori of the unperturbed motion (drift surfaces).

Now we introduce the action-angle canonical variables of the unperturbed system[2, 1]. The canonical angular coordinates associated with a drift surface, θ and ϕ, are defined as follows:

$$\theta(\vartheta, J, \mathcal{E}) = \omega_b \int_0^\vartheta d\vartheta/\dot{\vartheta}, \quad \phi(\varphi, \vartheta, J, \mathcal{E}) = \varphi + \chi(\vartheta, J, \mathcal{E}), \tag{2}$$

where $\chi \equiv \int_0^\vartheta d\vartheta\,(\omega_\phi - \dot{\varphi})/\dot{\vartheta}$ is a continuous periodic function of ϑ, the integrals are taken along the orbit $J(r, \vartheta) = \text{const}$, $\dot{\vartheta}(\vartheta, J, \mathcal{E})$ and $\dot{\varphi}(\vartheta, J, \mathcal{E})$ are determined by the Euler-Lagrange equations following from Eq. (1), ω_ϕ and ω_b are the frequencies of the toroidal and poloidal motion, respectively. The corresponding action variables are J and

$$J_p(J, \mathcal{E}) = \frac{1}{2\pi} \oint d\vartheta\ \frac{e}{c} F(r). \tag{3}$$

One can show that the Lagrangian (1) can be transformed into the form

$$\mathcal{L}_u = J_p(J, \mathcal{E})\dot{\theta} + J\dot{\phi} - \mathcal{E}. \tag{4}$$

3. The motion in the presence of a perturbation

We consider a slowly varying perturbation of the electromagnetic field characterized by the vector potential $\varepsilon\vec{A}(r, \vartheta, n\varphi + \omega t, \varepsilon t)$ and the scalar potential $\varepsilon\Phi(r, \vartheta, n\varphi + \omega t, \varepsilon t)$, where ω is the perturbation frequency, n is the toroidal wavenumber, and ε is the ordering parameter (we will set $\varepsilon = 1$ to obtain physically meaningful results). The particles on a drift surface are in resonance with such a perturbation if the frequencies of their motion satisfy the condition $\omega + n\omega_\phi - s\omega_b = 0$. To study the motion near this resonance, we introduce new canonical angles associated with the resonance, $\hat{\phi} = \omega t + n\phi - s\theta$ and $\hat{\theta} = \theta/n$. Then the Lagrangian (4) takes the following form:

$$\mathcal{L}_u = \hat{J}_p(\hat{J}, H)\dot{\hat{\theta}} + \hat{J}\dot{\hat{\phi}} - H, \tag{5}$$

where $\hat{J}_p \equiv nJ_p + sJ$, $\hat{J} \equiv J/n$, $H = \mathcal{E} + \omega\hat{J}$. An immediate consequence of the Euler-Lagrange equations resulting from Eq. (5) is that

$$\left.\frac{\partial \hat{J}_p}{\partial \hat{J}}\right|_{\hat{J}=\hat{J}_r(H)} = 0, \tag{6}$$

where $\hat{J}_r(H)$ is the value of $\hat{J}$ at the resonant drift surface.

We present $\mathcal{L}_u$ as a sum of two components of different orders. The first of them,

$$\mathcal{L}^{(0)} = \hat{J}_p^{(0)}(H)\dot{\hat{\theta}} + \hat{J}\dot{\hat{\phi}} - H, \tag{7}$$

where $\hat{J}_p^{(0)}(H) \equiv \hat{J}_p(H, \hat{J}_r(H))$, describes the motion in resonance with the wave, $\dot{\hat{\phi}} = 0$. The residual $\hat{J}_p^{(1)} \equiv \hat{J}_p(H, \hat{J}) - \hat{J}_p^{(0)}(H)$ responsible for "detuning" from the resonance is assumed to be of the order of ε. Thus, the guiding centre Lagrangian [3] takes in the presence of the perturbation the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \varepsilon \left(\hat{J}_p^{(1)}(H, \hat{J})\dot{\hat{\theta}} + \frac{e}{c}\vec{A} \cdot \dot{\vec{x}} - e\Phi \right). \tag{8}$$

To perform the bounce averaging, we apply the Hamiltonian perturbation theory in non-canonical co-ordinates [4]. We look for a sequence of co-ordinate transformations of the form $\vec{z'} = ... \exp(\varepsilon^2 \vec{G}^{(2)}) \exp(\varepsilon \vec{G}^{(1)})\vec{z}$, which makes the Lagrangian $\mathcal{L}$ independent of $\hat{\theta}$, where $\vec{z} = (\hat{\theta}, \hat{\phi}, H, \hat{J}, t)$ is the vector of the variables, and $\vec{G}^{(j)}$, $j = 1, 2, ...$, are the generators of the transformations. In addition, we demand that the form of the final Lagrangian be the same as the form of $\mathcal{L}_u$ [Eq. (5)]. Then we expand the transformed Lagrangian in powers of ε (see [4] for details). Solving the resulting algebraic equations for $\vec{G}^{(j)}$ successively order by order, we arrive at the Lagrangian

$$\mathcal{L} = I(H', J', \phi')\dot{\theta}' + J'\dot{\phi}' - H' \tag{9}$$

Here, through first order, $H' = H + \varepsilon e \langle \Phi \rangle$ is the perturbed Hamiltonian, $\left\langle J' - \hat{J} \right\rangle = \left\langle \phi' - \hat{\phi} \right\rangle = \left\langle \theta' - \hat{\theta} \right\rangle = 0$, where $\langle (...) \rangle \equiv \oint d\hat{\theta}\, (...) \,/(2\pi)$ denotes bounce averaging, and $I(H', J', \phi') = \bar{I}(H', J') + \tilde{I}(H', J', \phi')$ is a constant of the bounce-averaged motion (adiabatic invariant) (cf. Ref. [1]). Its phase-independent part, $\bar{I}$, coincides with $\hat{J}_p$, and the oscillating part associated with the perturbation is given by

$$\tilde{I} = -\frac{e}{\Omega_r}\langle \Phi \rangle + \frac{e}{c}\left\langle \vec{A} \cdot \frac{\partial \vec{x}}{\partial \hat{\theta}} \right\rangle, \tag{10}$$

where $\Omega_r \equiv (d\hat{J}^{(0)}/dH)^{-1} = \omega_b(H, \hat{J}_r(H))/n$.

The bounce-averaged equations of motion immediately result from Eq. (9):

$$\dot{\phi}' = -\Omega \left(\frac{\partial I}{\partial J'} \right), \quad \dot{J}' = \Omega \left(\frac{\partial I}{\partial \phi'} \right), \quad \dot{H}' = -\Omega \left(\frac{\partial I}{\partial t} \right), \tag{11}$$

where $\Omega \equiv (\partial I / \partial H')^{-1} = \omega_b / n$.

4. Resonance islands (superbanana orbits).

Let us consider the case of a steady-state rotating perturbation, when $\vec{A}$ and Φ do not depend on the slow time εt. The oscillating part part of I can be presented as $\tilde{I} = \Delta I \exp(i\hat{\phi})$, where ΔI is calculated by integration of the perturbation along the unperturbed trajectories, see Eq. (10). Following [1, 6], we expand I in a power series in $J - J_r$, keeping the terms of $\bar{I}$ up to the second order and only the zero-order term of $\tilde{I}$ and using Eq. (6). This yields:

$$I = \bar{I}(J_r) + \frac{1}{2} \left. \frac{\partial^2 \bar{I}}{\partial J^2} \right|_{J=J_r} (J - J_r)^2 + \left. \Delta I \right|_{J=J_r} \exp(i\hat{\phi}). \tag{12}$$

The lines of level of I represent orbits of the bounce-averaged motion in the presence of the perturbation in the vicinity of the resonance. They form a Chirikov island near the resonance[6]. The separatrix of the island separates the orbits of the particles that are trapped with respect to the perturbation of the electromagnetic field (i.e., superbanana orbits) from the other orbits. The equation (12) enables us to to evaluate the island width ΔJ (i.e., the maximum superbanana orbit width) in terms of the amplitude of $\tilde{I}$ on the resonant drift surface, ΔI:

$$\Delta J = 2\,|\Delta I|^{1/2}\left|n\frac{\partial}{\partial J}\frac{\omega_\phi}{\omega_b}\right|^{-1/2}. \tag{13}$$

Substituting Eq. (12) to Eqs. (11), we can find the period of the superbanana motion. Near the island centre, it is given by

$$\tau_{sb} = \frac{2\pi}{\omega_b}\left|n\Delta I\frac{\partial}{\partial J}\frac{\omega_\phi}{\omega_b}\right|^{-1/2} \tag{14}$$

5. Discussion of the results

The noncanonical Hamiltonian perturbation technique [4] has enabled us to obtain an adiabatic invariant and equations of the motion of a fast ion near a resonance. Although the results are presented here only through first order in ε, the calculations are easily extended to higher orders. One can expect that the obtained invariant can be calculated analytically only in special cases. It seems more practical to calculate the invariant numerically. As shown in Ref. [1], such a procedure allows to find the resonance island width by means of calculating several integrals along an unperturbed resonance orbit, which is much less time-consuming than generating a Poincaré map. The proposed approach is rather general and may be used for studying the resonant motion of particles in other axisymmetric systems and quasisymmetric stellarators.

Acknowledgements. The research described in this publication was made possible in part by the Research Contract No. 8924 of the International Atomic Energy Agency (IAEA) and Award No. UP2-290 of the Government of Ukraine and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union.

References

[1] Kolesnichenko Ya.I. et al.: “Theory of resonance influence of sawtooth crashes on ions with large orbit width”. Report PPPL-3286 (1998); Phys. Plasmas, *in press*.

[2] Kaufman A.N.: Phys. Fluids **15**, 1063 (1972).

[3] Littlejohn R.G.: Phys. Scr. **T2/1**, 119 (1982).

[4] Littlejohn R.G.: J. Math. Phys. **25**, 742 (1982).

[5] White R.B., et al.: Phys. Fluids **25**, 575 (1982).

[6] Chirikov B.V.: Phys. Reports **52**, 265 (1979).