BALLOONING EIGENMODE STRUCTURE OF LOW-SHEAR STELLARATORS

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1. Introduction

The local eigenvalue of the ballooning equation as a function of magnetic surface label ψ, field line label α and the poloidal-angle-like parameter θ_k can be regarded as a dispersion relation giving local knowledge of the global structure of flute-like instabilities in general toroidal confinement devices [1] via a semiclassical quantization technique. This approach has been successfully applied to interchange instabilities in a torsatron/heliotron test case [2], where the global magnetic shear is high, with strong θ_k dependence and weak field-line dependence. The present paper addresses the situation in a different class of stellarator where the global magnetic shear is weak. In this case the θ_k dependence is weak and the field-line dependence is strong. We present the results of a numerical study of an unstable equilibrium of the H-1 heliac, giving a qualitative classification scheme for the ballooning eigenvalue branches based on symmetry properties of the ballooning equation in the zero-shear limit.

2. Ballooning equation

As in [1] we write the magnetic field as $\mathrm{B} = \boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi - q\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\psi \equiv \boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\psi$, where the *field-line label* $\alpha \equiv \zeta - q\theta$, with θ and ζ being the poloidal and toroidal angles, respectively, and $q(\psi)$ the inverse of the rotational transform.

The ballooning equation in the incompressible approximation [3] is defined on a field line, $\alpha = \mathrm{const}$,

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\,\mathcal{A}\,\frac{\mathrm{d}}{\mathrm{d}\theta} - \mathcal{K} + \mathcal{N}\omega^2\rho\right]\xi = 0\ , \tag{1}$$

where ξ is proportional to the normal displacement of a perturbed field line and $\mathrm{d}/\,\mathrm{d}\theta$ is the total θ-derivative along the field line (i.e. at fixed α). The coefficient of the second derivative term is

$$\mathcal{A} = \frac{1}{J|\boldsymbol{\nabla}\psi|^2} + \frac{|\boldsymbol{\nabla}\psi|^2}{JB^2}\left[\mathcal{R} + q'(\psi)\,(\theta-\theta_k)\right]^2\ , \tag{2}$$

the normalizing factor $\mathcal{N} = J^2\mathcal{A}$, with $J \equiv (\boldsymbol{\nabla}\psi\boldsymbol{\cdot}\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta)^{-1}$ being the Jacobian, and

$$\mathcal{K} = \frac{-2Jp'(\psi)}{|\boldsymbol{\nabla}\psi|}\left(\kappa_{\mathrm{n}} + \frac{|\boldsymbol{\nabla}\psi|^2}{B}\left[\mathcal{R} + q'(\psi)\,(\theta-\theta_k)\right]\,\kappa_{\mathrm{g}}\right)\ , \tag{3}$$

which contains the potentially destabilizing effect of the pressure gradient p'. The normal and geodesic curvatures are defined in terms of the field-line curvature, $\boldsymbol{\kappa}$, by $\kappa_{\mathrm{n}} \equiv \boldsymbol{\kappa}\boldsymbol{\cdot}\boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|$

and $\kappa_g \equiv \boldsymbol{\kappa}\boldsymbol{\cdot}\boldsymbol{\nabla}\psi\times\mathbf{B}/|B\boldsymbol{\nabla}\psi|$ respectively. The *integrated residual shear* [4] is given by $\mathcal{R} = (q\boldsymbol{\nabla}\psi\boldsymbol{\cdot}\boldsymbol{\nabla}\theta - \boldsymbol{\nabla}\psi\boldsymbol{\cdot}\boldsymbol{\nabla}\zeta)/|\boldsymbol{\nabla}\psi|^2$.

These coefficients consist of products of periodic equilibrium quantities and *secular terms* involving powers of $q'(\psi)(\theta - \theta_k)$. **Note:** The equilibrium quantities have periods 2π in θ and $2\pi/M$ in ζ, where M is the number of field periods. However, when restricted to the field line $\zeta = \alpha + q\theta$, they are not periodic unless q is a rational fraction — in general they are *quasiperiodic* functions of θ.

The ballooning equation is an ODE eigenvalue problem to be solved for the eigenvalue $\rho\omega^2$ under the boundary conditions that ξ vanishes as $\theta \to \pm\infty$. The constant term θ_k determines the direction of the wave vector in the WKB formalism [1], and thus to determine the global eigenvalues by ray tracing [2] we need to know the detailed dependence of the eigenvalue on all three parameters, ψ, α and θ_k. This paper concerns only the problem of understanding this dependence and does not concern itself further with the ray tracing problem.

3. Low shear study

A three-dimensional plasma equilibrium in a three-fold symmetric ($M = 3$) helical-axis stellarator, the H-1 heliac [5], was calculated numerically using the preconditioned VMEC equilibrium code [6]. The case studied, which uses the "standard" vacuum magnetic field configuration of H-1, has a volume-averaged β of 1% and a safety factor q which varies by less than 5% over the whole of the plasma volume. The pressure profile was constructed by taking the case constructed by Cooper and Gardner [7], which is marginally ballooning stable over the whole plasma, and then increasing the pressure by a constant factor so that the equilibrium becomes ballooning unstable.

VMEC uses $s \equiv \phi(\psi)/\phi(\psi_a)$ instead of ψ as the magnetic surface label, where $2\pi\phi$ is the toroidal magnetic flux and s has been normalized to unity at the plasma edge $\psi = \psi_a$.

The incompressible ballooning equation was solved on a 124×400 lattice in the (s, α) parameter space covering one field period, $0 \leq \alpha < 2\pi/3$, for the case $\theta_k = 0$. The ballooning equation eq. (1) was solved numerically by a shooting method using a grid containing 300 points for each 2π circuit in θ, for a total of 50 circuits. The eigenvalue was iterated upon until the boundary conditions were satisfied.

A contour plot of the ballooning eigenvalue in s, α space is shown in Fig. 1. It is seen that there are distinct branches, localized around field lines that change with s, except for the branches with maximum growth rate at α an integer multiple of $2\pi/3$. Some typical eigenfunctions at $s = 0.70$, $q = 0.88$, are shown in Fig. 2, for $\alpha = 0$ (marked by a '$\Box$' in Fig. 1), $\alpha = 0.39$ (this point being the crossover point of two branches, marked by an '$\times$' in Fig. 1), and $\alpha = 0.854$ (marked by a '$\triangle$' in Fig. 1), which is the most unstable point for this value of s on the branch to the left of the centre line in Fig. 1. In each case the eigenfunctions peak near $\theta = 2\pi n$ where n is an integer: $n = 0$ for the '$\Box$' case, $n = -2$ or 3 for the '$\times$' case, and $n = 1$ for the '$\triangle$' case.

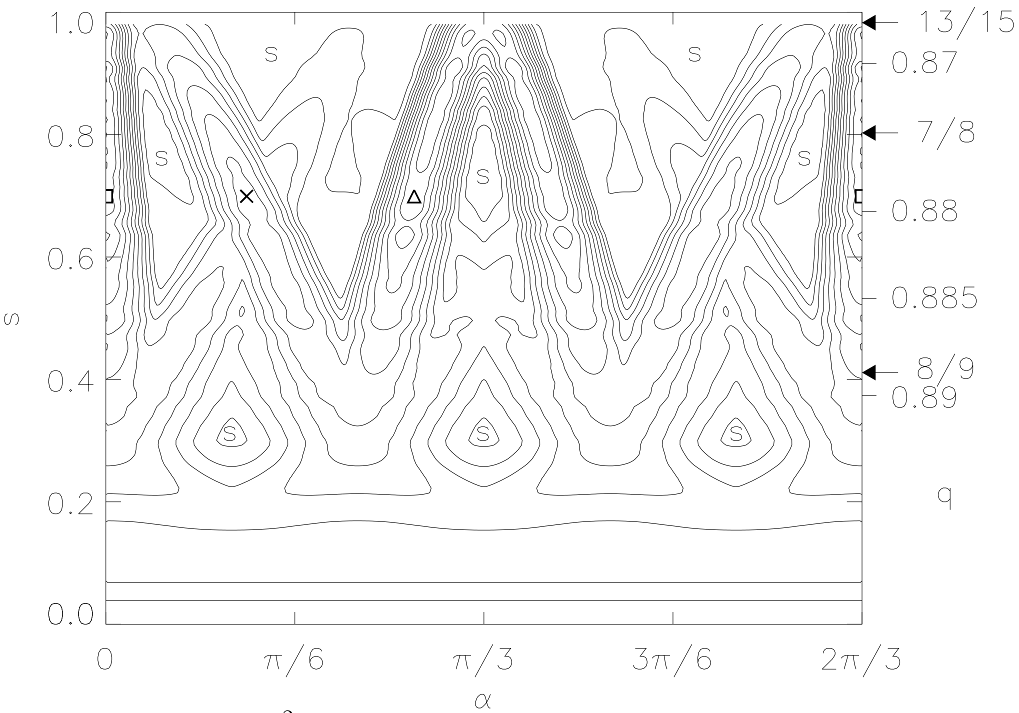

Fig. 1: Contours of $\rho\omega^2$ in the field-line label α and surface label s plane at $\theta_k = 0$.

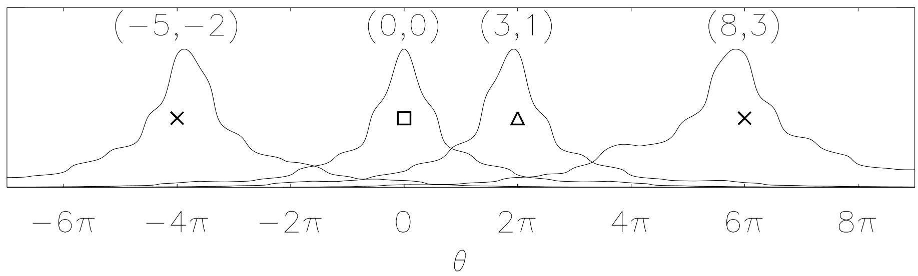

Fig. 2: Eigenfunctions at the points marked by $\times$ etc. in Fig. 1.

4. Classification of branches

We now show that all these branches can be related by the application of an approximate discrete symmetry operation that becomes exact in the limit of zero shear. This is the symmetry operation of complete poloidal rotation

$$P_{\text{eqm}}: \quad \theta \mapsto \theta + 2\pi \; , \; \alpha \mapsto \alpha - 2\pi q \; , \tag{4}$$

which is an exact symmetry of the equilibrium, but only an approximate symmetry operation of the ballooning equation because θ_k is taken to be fixed, in contrast with the *full* poloidal symmetry operation for eq.(1)

$$P_{\text{bal}}: \quad \theta \mapsto \theta + 2\pi \; , \; \alpha \mapsto \alpha - 2\pi q \; , \; \theta_k \mapsto \theta_k + 2\pi \; , \tag{5}$$

which is an exact symmetry of the ballooning equation because both the periodic equilibrium terms and the secular terms involving $(\theta - \theta_k)q'$ are invariant under P_{bal}.

On the other hand, the ballooning eigenvalue is afilnot in general invariant under P_{eqm} because $P_{\text{eqm}}(\theta - \theta_k) = \theta - \theta_k + 2\pi$. However, in the zero-shear limit $q' \to 0$, the secular terms in $(\theta - \theta_k)q'$ disappear from the equation and it becomes invariant. Thus the operation $P_{\rm eqm}$ is an *approximate* symmetry of the ballooning equation for low-global-shear systems.

To relate the branches seen in Fig. 1 with each other we also need to introduce the toroidal symmetry operation of rotation through a field period

$$T_M : \theta \mapsto \theta \ , \ \alpha \mapsto \alpha + 2\pi/M \ , \ \theta_k \mapsto \theta_k \ , \tag{6}$$

which is an exact symmetry of the ballooning equation.

Now we consider repeated applications of the exact symmetry operation T_M and the approximate symmetry operation $P_{\rm eqm}$. Under the operation $T_M^m P_{\rm eqm}^n$, $\alpha \mapsto \alpha + 2\pi(m/M - nq)$ and $\theta \mapsto \theta + 2\pi n$. Thus, if $\xi^{(0,0)}(\theta)$ is an eigenfunction on the field line $\alpha^{(0,0)}$, then, in a low-shear system, $\xi^{(m,n)} \equiv \xi^{(0,0)}(\theta - 2\pi n)$ is an approximate eigenfunction on the field line

$$\alpha^{(m,n)} \equiv \alpha^{(0,0)} + 2\pi \left(\frac{m}{M} - nq \right) \ . \tag{7}$$

As the reference case $(0,0)$ we choose the branch with maximum growth rate at $\alpha = \alpha^{(0,0)} \equiv 0$ independent of s (see Fig. 1). We then identify the branches depicted in Fig. 2 as (from left to right) $(-5,-2)$, $(0,0)$, $(3,1)$ and $(8,3)$, which, from eq.(7), would peak on the field lines $\alpha^{(-5,-2)} \approx 0.57$, $\alpha^{(3,1)} \approx 0.76$ and $\alpha^{(8,3)} \approx 0.19$ if the symmetry were exact.

Comparing the predicted locations with the numerically observed locations of the peak growth rates we see that the agreement is far from perfect, indicating that the symmetry-breaking effect of global magnetic shear not only changes the growth rate but shifts the peak as well.

The identifications of the branches is confirmed by calculating the s-dependence of the loci of maximum growth rate. For instance eq.(7) predicts that the loci of maximum growth rate for the $(0,0)$ mode and the $(8,3)$ mode cross at $q = 8/9$, as do the corresponding loci for the $(-5,-2)$ and $(3,1)$ modes. This value of q occurs at $s \approx 0.41$. Comparing with Fig. 1 we see that the predicted crossovers do indeed occur, and at approximately the predicted value of s.

Acknowledgements

The ballooning calculations were carried out on the Australian National University Supercomputer Facility's Fujitsu VPP300 vector processor. We thank Dr. Henry Gardner for providing the H-1 heliac VMEC input files and Dr. S. P. Hirshman for use of the VMEC equilibrium code.

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