CATASTROPHE-TYPE MODELS TO FIT NON-LINEAR PLASMA RESPONSE FUNCTIONS

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Abstract In this paper, first results of a statistical approach are presented to fit catastrophe-type response surfaces to plasma energy confinement time data. Such surfaces can reflect the stored energy and at the same time the accessibility region of H-mode ('power threshold' and other operational boundaries). The method, outlined in [1], is therefore to be viewed as a unification of regression and discriminant analysis. Actual implementation required the development of a suitable statistical fitting algorithm. A parallel FORTRAN-90 code in a data-analytic environment has been created to carry out practical analyses. Results on experimental L and H-mode data from JET and DIII-D actually show a third order cusp-type regression surface of the normalised plasma energy as a function of input power and plasma density and indicate the possible fruitfulness to pursue this line of approach.

Regression and Discriminant Analysis First principle theoretical analysis of thermonuclear transport, despite the multifarious approaches that have been taken, has hitherto met with only partial success in the quantitative prediction of the confinement time. Therefore, empirical transport scaling studies have been actively pursued since early work by Goldston [2] and others, especially in relation to the collaborative assemblage of international databases stimulated by the ITER project [3,4,5,6]. The transformation to log scale finds its practical justification in an approximately homogeneous residual scatter (homoskedasticity) on this scale [7]. However, log-linear models (i.e. simple power laws) to describe the response surface are known not to reflect accurately the underlying physics and are to be used only as a first approximation to the regression surface. Some significant systematic deviations ('curvatures') from such simple empirical models have been found in the form of interaction models [8,6,9] and of two-term power law scalings [10,11], a generalisation of offset-linear scalings. As summarised in [3], these non-linear models have led to the presently adopted 95% interval estimate [3.5,8] s. for ITER, which is about twice as large as the interval would be under the hypothesis that log-linear functional forms are sufficient for extrapolation.

Almost in parallel to the confinement scaling problem, the issue has been addressed in which region of plasma (control) parameter space (ELMy) H-mode can be produced. As a systematic method, the use of discriminant analysis has been suggested in [12] and is trying to find its way into more recent analyses of the international threshold database [13,14]. In its usual form, the regression problem is reversed: given two groups of shots (e.g. L-mode, '0', and ELMy H-mode of some type, '1'), find the region in control parameter space where each of these two types of shots are (expected to be) producible, based on historical data. The regions are separated by discriminating boundaries, the predictive performance of which can be analysed by ROC curve analysis [15]. Log-linear discriminant analysis has much in common with log-linear regression analysis. To estimate log non-linear boundaries, some type of natural border seems to have been been reached with quadratic discriminant analysis on the one side and non-parametric analysis restricted by a multiplicative independence assumption on the other [12].

The 'shifting nature' of log-linear confinement time scalings when datasets are updated and the presence of several non-linear curvatures, some of them leading to substantial lower ITER confinement than the log-linear point predictions (as suggested by [9] and summarized in [3]), provided motivation to pursue the approach of 'catastrophe fitting'. Here, multi-dimensional folded regression surfaces are fitted which can be viewed, in a number of cases, as the stationary points of non-linear dynamical systems $\dot{\eta} = -\partial V(\eta, \underline{\xi}\,\underline{\zeta})/\partial\eta$, where $\eta = g_{\underline{\mu}}(W_{th})$ is the response variable, and $\underline{\xi} = \underline{f}_{\underline{\lambda}}(P_{L'}, \ldots)$ are the primary control parameters and $\underline{\zeta} = \underline{h}_{\underline{\nu}}(\overline{n}_e, B_t, \ldots)$ are the bifurcation parameters. The singularities of such surfaces have been coined as catastrophic points by Thom [16] to indicate that the qualitative behaviour of the non-linear differential equation changes drastically for small changes of the governing parameters ($\underline{\lambda}, \underline{\mu}, \underline{\nu}$). Bifurcation phenomena abound in several varieties in L-H transition theory [17]. While practical fitting of such type of multi-dimensional regression surfaces, in the metric of the measurement errors, has until now been hampered by the absence of both statistical software and pertinent regression methodology, we are now able to report here some progress in this direction.

Method The method, devised for the above purpose and implemented in the program CATA-1, can in simple terms be described as projecting the data in $\mathbb{R}^p$ on the regression surface and using a global natural selection algorithm to search for the best fitting surface. A parallel implementation has been made in FORTRAN-90 on the CRAY T3E with routines from NAG and an input/output jacket on a workstation in Mathematica and S-PLUS/SAS.

Results In the figures below, we present the results from two analyses to illustrate the method. In the column, the JET ($\langle T\rangle, P_{L'}, \overline{n}_e$) measurements from the standard DB2 L-mode and DB3 H-mode datasets [18,3,4,5], are plotted ($\langle T\rangle \sim W_{th}/\overline{n}_e V$ against $P_{L'}$ for slices of $\overline{n}_e$) for D into D only. Symbols 0, 1 and 2 are used to denote L-mode, ELM-free H-mode (using $0.85W_{th}$) and ELMy H-mode, respectively. These data have been fitted using a function describing an elementary cusp around a critical point (x_0, y_0, z_0) in $\mathbb{R}^3$

$$S_1 : a_{y3}(y - y_3)^3 + a_{yz}(y - y_0)(z - z_0) - (x - x_0) = 0 \tag{1}$$

where $x \equiv \ln P_{L'}$, $y \equiv \ln\langle T\rangle$ and $z \equiv \ln \overline{n}_e$. The coefficients (a_{y3}, a_{yz}) and (x_0, y_0, y_3, z_0) are regression parameters determined by the data, including the measurement errors, which were assumed to be (15%, 10%, 5%) for ($\langle T\rangle$, $P_{L'}$, $\overline{n}_e$). The fitted values were (1.8,-1.6) and (1.2,0.8,0.25,1.15). The meta-stable branch of the surface (where $\partial x(y,z)/\partial y < 0$) has been excluded from the regression. According to this first analysis, the data suggest the presence of a two-fold critical point (at which the LH and HL transition lines merge and L-mode becomes 'as good as H-mode') at low density, the physical or technical reason of which has to be clarified. Using W_{th}/I_P instead of the temperature $\langle T\rangle$ gives somewhat similar results. It is obvious that the dependence $\ln P_{L'} \sim \ln^3\langle T\rangle$ is too strong far from the critical region. One way to make a better transition between the local catastrophe behaviour and a global fit is to use a polynomial expression on normal scale. However, this requires the regression to be performed under heteroskedastic errors, a feature at present not yet implemented in the program. Therefore, in the second example another approach was taken. Using the DIII-D subset of the released databases (L-mode and H-mode, D into D only, $-0.05 < \dot{W}_{th}/P_{L'} < 0.2$)) the following surface was fitted.

$$\begin{aligned} S_2 : & [a_{y3}(y-y_0)^3 + a_{yz}(y-y_0)(z-z_0)]f_s(y) \\ & +[a_{y,1}(y-y0) + a_{y,2}|y-y_0| + b_z(z-z_0)](1-f_s(y)) = x - x_0 \end{aligned} \tag{2}$$

Here $f_s(y) = (1 + (\frac{y-y_0}{s})^6)^{-1}$ is a smooth function (with adjustable scale factor s) to model the transition between the region near y_0 (where f_s is unity) and the region far from y_0 (where f_s is zero). The point (x_0, y_0, z_0) is again a triple point, and $(a_{y3}, a_{y,1}, a_{y,2}, a_{yz}, b_z)$ are parameters determining the local slope of the regression surface. While using engineering units and natural logarithms for W_{th}/I_p, $P_{L'}$ and $\overline{n}_{e,19}$, minimisation of the residual error (in the metric of the measurement errors) yielded the values (1.45) for s, $(1.5, -1.0, 1.6)$ for (x_0, y_0, z_0) and $(3.0, 2.0, 3.0, -0.75, 2.2)$ for the above five slope parameters. The corresponding surface (plotted with Mathematica) is shown below. As for JET, it suggests a merging between H-mode and L-mode at low densities. Note that the terms with $a_{y,1}$ and $a_{y,2}$ allow a different log-linear power dependence for L and H-mode far from the critical point and that a log-linear type model is included in S_2. On the other hand, the model imposed allowed only one cusp region, while for high densities ($\overline{n}_{e,19} > 7$) only H-mode points were available. One can easily derive that both the L to H and the H to L transition lines are defined by $\partial x(y,z)/\partial y = 0$ while the coefficient b_z determines the slope between the input power and the density (on log scale) that are needed to achieve a fixed temperature. Hence, the regression surface indeed gives a unified description of the confinement and the power threshold scaling.

Conclusion A method has been developed and implemented in a computer program ('CATA1') to fit various classes of catastrophe-type regression surfaces. In this way it is possible to analyse empirically L- and H-mode confinement as well as the power threshold problem in a more unified way than hitherto.

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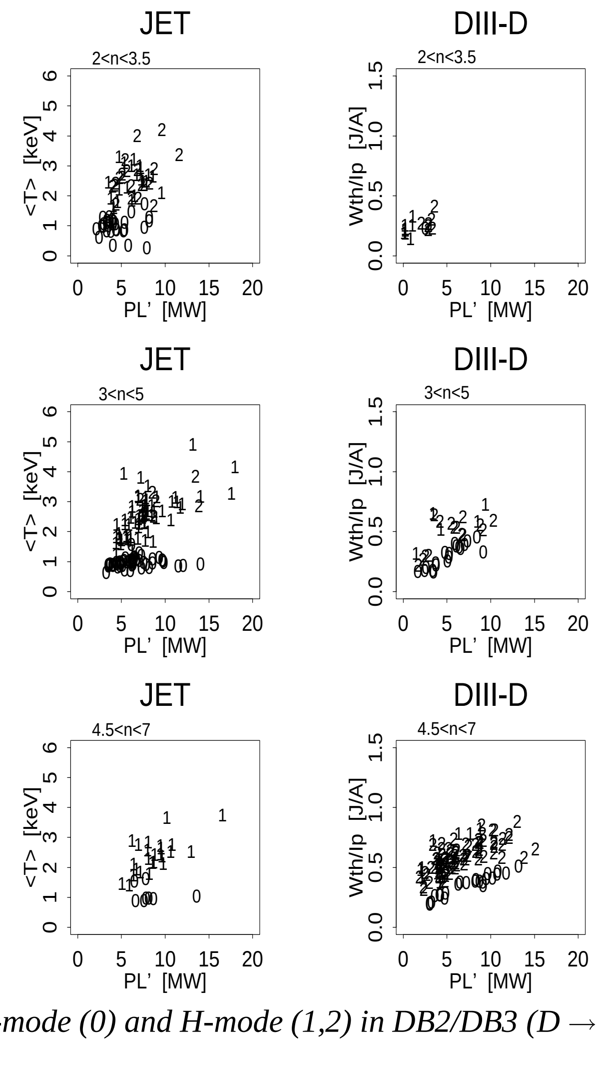
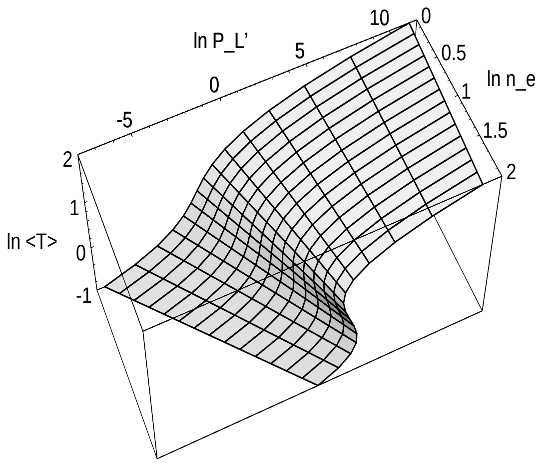
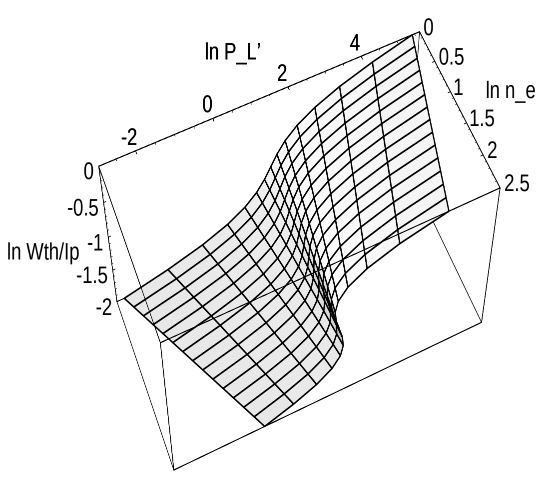


L-mode (0) and H-mode (1,2) in DB2/DB3 (D → D)



Fitted surface for $\langle T\rangle \sim W_{th}/\overline{n}_e V$ (JET)



Fitted surface for W_{th}/I_p (DIII-D)