FLUCTUATIONS IN PLASMAS WITH TIME-NONLOCAL (NON-MARKOVIAN) DIFFUSION

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1. Introduction

Recently, much interest has been attracted to the studies of physical systems with statistical properties which can not be described within the concept of the Markovian random processes. This concerns anomalous transport in turbulent plasmas [1-3], "strange" spatial diffusion of magnetic field lines [2], Brownian motion of macroparticles in complex fluids [4,5]. etc. One of the most important problem here is to describe the kinetic properties of such systems and to understand the role of the memory effects in transport phenomena. Obviously, studies of electromagnetic fluctuations in the relevant systems provide a natural basis for the description of kinetic properties, since kinetic coefficients can be expressed in terms of correlation functions of electric field and particle density fluctuations. On the other hand, fluctuations themselves strongly depend on the transport processes. Thus, the problem arises to describe selfconsistently both fluctuation spectra and kinetic coefficients. The purpose of this paper is to work out a theory of potential electric fluctuations in turbulent plasmas with regard for the memory effects associated with large scale turbulent perturbations, to calculate the kinetic coefficients, and to estimate the diffusion level for saturated turbulence.

2. Transition Probability Approach to the Theory of Turbulent Plasmas

We start from the equation for the microscopic phase density

$$N(X,t) = \frac{1}{n}\sum_{i=1}^{N}\delta(X - X_i(t)), \qquad X = (\mathbf{r},\mathbf{v}) \tag{1}$$

which is given by

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \frac{e}{m}\mathbf{E}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{v}}\right\} N(X,t) = 0. \tag{2}$$

Here $\mathbf{E}(\mathbf{r},t)$ is the microscopic electric field. The kinetic equation for the distribution function $f(X,t) = < N(X,t) >$ can be obtained from Eq. (2) by its statistical averaging over the Gibbs ensemble, corresponding to the time-averaging over physically infinitesimal time

$$L^0 f(X,t) \equiv \left\{\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \frac{e}{m} < \mathbf{E}(\mathbf{r},t) > \frac{\partial}{\partial \mathbf{v}}\right\} f(X,t) = I, \tag{3}$$

where

$$I = -\frac{e}{m}\frac{\partial}{\partial \mathbf{v}} < \delta\mathbf{E}(\mathbf{r},t)\delta N(X,t) > . \tag{4}$$

The equation for fluctuations reduces then to

$$\left\{\hat{L}^0 + \frac{e}{m}\delta\mathbf{E}(\mathbf{r},t)\frac{\partial}{\partial\mathbf{v}}\right\}\delta N(X,t) = -\frac{e}{m}\delta\mathbf{E}\frac{\partial f}{\partial\mathbf{v}}, \tag{5}$$

The formal solution of Eq. (5) is given by

$$\delta N(X,T) = \delta N^0(X,t) - \frac{e}{m}\int\limits_0^t dt' \int dX' W_m(X,X';t,t')\delta\mathbf{E}(\mathbf{r}',t')\frac{\partial f(X',t)}{\partial t}. \tag{6}$$

Here $\delta N^{(0)}(X,t)$ is the fluctuation of the microscopic phase density due to direct particle transitions which satisfies the equation

$$\left\{\hat{L}^{(0)} + \frac{e}{m}\delta\mathbf{E}(\mathbf{r},t)\frac{\partial}{\partial\mathbf{v}}\right\}\delta N^0(X,t) = 0, \tag{7}$$

$W_m(X,X';t,t')$ is the "microscopic" probability of a particle transition from the point X' to X during the time interval $t-t'$. Obviously the equation for such probability is

$$\left\{\hat{L}^{(0)} + \frac{e}{m}\delta\mathbf{E}(\mathbf{r},t)\frac{\partial}{\partial\mathbf{v}}\right\}W_m(X,X';t,t') = 0 \tag{8}$$

with the initial condition $W_m(X,X';t,t') = \delta(X-X')$. The solution of Eq. (8) is

$$W_m(X,X';t,t') = \delta(X - X' - \Delta X(X',t';t)), \tag{9}$$

where $\Delta X(X',t';t)$ is the particle phase displacement in course of its motion in the microfields.

We now assume that the distribution function slowly changes within the spatial and velocity fluctuation scales. Then, combining Eqs. (4), (6) and (8) it is possible to derive the kinetic equation with non-Markovian collision term, i.e.,

$$\hat{L}^{(0)}f(X,t) = \frac{\partial}{\partial v_i}\int\limits_0^t dt' \left\{b_i(t,t'v)f(X,t') + \frac{\partial}{\partial v_j}\left[\tilde{D}_{ij}^{(v)}(t,t',v)f(X,t')\right]\right\}, \tag{10}$$

where

$$\begin{aligned}
b_i(t,t',v) &= b_i^{(1)}(t,t',\mathbf{v}) + b_i^{(2)}(t,t',\mathbf{v}) \\
b_i^{(1)}(t,t',\mathbf{v}) &= \frac{8\pi e^2 n}{m}\int\frac{d\omega}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_i < W_{m\mathbf{k}\omega}(\mathbf{v}) >}{k^2\varepsilon(k,\omega)}\delta(t-t') \\
b_i^{(2)}(t,t',v) &= -\frac{\partial}{\partial\mathbf{v}}\left(\frac{e}{m}\right)^2\int\frac{d\mathbf{k}}{(2\pi)^3} < \delta E_i(t)\delta E_j(t') >_{\mathbf{k}} < W_{m\mathbf{k}}(v,t,t') > \\
\tilde{D}_{ij}^{v}(t,t'v) &= \left(\frac{e}{m}\right)^2\int\frac{d\mathbf{k}}{(2\pi)^3} < \delta E_i(t)\delta E_j(t') >_{\mathbf{k}} < W_{m\mathbf{k}}(\mathbf{v},t,t') > \\
\varepsilon(k,\omega) &= 1 - \frac{4\pi e^2 n}{m}\int d\mathbf{v} < W_{m\mathbf{k}\omega}(\mathbf{v}) > \mathbf{k}\frac{\partial f}{\partial\mathbf{v}}
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\langle W_{m\mathbf{k}}(\mathbf{v},t,t')\rangle &= \int d\Delta X e^{-i\mathbf{k}\Delta\mathbf{r}}\langle W(X+\Delta X, X; t,t')\rangle \\
\langle W_{m\mathbf{k}\omega}(\mathbf{v})\rangle &= \int\limits_0^\infty d\tau e^{i\omega\tau}\langle W_{m\mathbf{k}}(\mathbf{v},t;t-\tau)\rangle.
\end{aligned} \tag{12}$$

We see that the collision integral is expressed in terms of the averaged transition probability $< W_m(X, X'; t, t') >$ into account the fluctuation field influence on the particle orbits.

Using Eq. (8), it is possible to show that this transition probability satisfies the equation of type (3), namely

$$\hat{L}^0 \langle W_m(X, X'; t, t') \rangle =$$

$$= \frac{\partial}{\partial v_i} \int_{t'}^{t} dt'' \left\{ b_i(t, t'') W(X, X'; t'', t') + \frac{\partial}{\partial v} \left[D^v_{ij}(t'', t', v) W(X, X'; t'', t') \right] \right\}. \qquad (13)$$

3. Electric Field Fluctuation Spectra

Using Eqs. (6), (8) we also find that

$$\langle \delta E^2 \rangle_{\mathbf{k}\omega} = \frac{16\pi^2 \langle \delta\rho^2 \rangle^{(0)}_{\mathbf{k}\omega}}{|\varepsilon(k, \omega)|^2}, \qquad (14)$$

where

$$\langle \delta\rho^2 \rangle^{(0)}_{\mathbf{k}\omega} = e^2 n \int d\mathbf{v} \int d\mathbf{v}' \langle \delta N^{(0)}(v) \delta N^{(0)}(v') \rangle_{\mathbf{k}\omega}. \qquad (15)$$

Taking into account Eq. (7), the latter relation can be rewritten as

$$\langle \delta\rho^2 \rangle^{(0)}_{\mathbf{k}\omega} = e^2 n \int d\mathbf{v} \left\{ f(X, t) \langle W_{mk\omega}(\mathbf{v}) \rangle + c.c. \right\}. \qquad (16)$$

We remind the reader that the quantities $\langle \delta\rho^2 \rangle^{(0)}_{\mathbf{k}\omega}$ and $\varepsilon(\mathbf{k}, \omega)$ as well as the distribution function f are time-dependent in the general case. So, Eqs (9) – (12) give the final solution of the problem just in the case when the time-dependence of the above mentioned quantities can be neglected.

4. Non-Markovian Transport and Saturation of Turbulence

The next problem is to solve the equation for the transition probability. In variuos particular cases its solutions can be found explicitely. Substituting these solutions into Eqs.(10) it is possible to derive self-consistent equations for the kinetic coefficients and thus to find the collision term. In particular, if the influence of fluctuation fields on particle trajectory can be neglected,

$$\langle W_{m\mathbf{k}}(\mathbf{v}, t, t') \rangle = e^{i\mathbf{k}\mathbf{v}(t-t')}$$

and Eq. (9) is reduced to the Balescu-Lenard equation provided the electric field correlations are determined by thermal particle motion (Eqs. (13), (15)). For electric field fluctuations of turbulent nature Eq. (9) reproduces the result of the quasilinear theory.

In the Markovian approximation ($< W(X, X'; t'', t') >$ in Eq. (13) is replaced by $< W(X, X'; t, t') >$) for the stationary one-dimensional case we have

$$< W_{m\mathbf{k}}(v, t, t - \tau) >= \exp \left[i\frac{kv}{\beta} \left(1 - e^{-\beta\tau} \right) - \frac{k^2}{\beta^2} \int_0^{\tau} d\xi D^v(t - \xi) \left(1 - e^{-\beta(\tau - \xi)} \right)^2 \right], \qquad (17)$$

where

$$D^v(t) = \int_0^t dt' \tilde{D}^v(t, t', v); \qquad \beta = \int_0^t dt' b(t, t', v).$$

In the stationary case for $\beta\tau \ll 1$, Eq.(17) reduces to

$$< W_{m\mathbf{k}}(v,t,t-\tau) >= \exp\left[ikv\tau - \frac{k^2 D^v \tau^3}{3}\right] \tag{18}$$

that leads to the results of the Dupree-Weinstock theory after substituting Eq. (18) into Eq. (10).

In the opposite limiting case, $\beta\tau \gg 1$, we obtain

$$< W_{m\mathbf{k}}(v,t,t-\tau) >= \exp(-k^2 D\tau) \tag{19}$$

that corresponds to the integration along the diffusive orbits. Here $D = D^v/\beta^2$ in the diffusion coefficient in the real space.

The solution of Eq. (13) in the non-Markovian one-dimensional case generates an equation for the Fourier-component of velocity diffusion coefficient, i.e.,

$$\tilde{D}^v_\omega = \left(\frac{e}{m}\right)^2 \int_0^\infty d\tau e^{i\omega\tau} \int \frac{dk}{2\pi} < \delta E(t)\delta E(t-\tau) >_k \int \frac{d\omega_1}{2\pi} e^{i\omega_i\tau} \int_0^\infty d\tau_1 e^{-i\omega\tau_1} \times \exp\left[\frac{ikv}{b_{\omega_1}} - \frac{k^2\tilde{D}^v_\omega}{b^2_\omega}\right]. \tag{20}$$

Here $\tilde{D}^v_\omega$ and b_ω are the Fourier-components of the quantities $\tilde{D}^v(t,t',v)$ and $b(t,t',v)$, respectively. This equation makes it possible to estimate the saturation level and diffusion coefficients for the turbulence generated by plasma instabilities. In particular, for the model turbulent spectrum,
$< \delta E(r,t)\delta E(r-R,t-\tau) >= \delta E_0^2 \cos(k_0 R - \omega_0\tau)$.
Eq. (20) yields a relation for the turbulent diffusion coefficient in the coordinate space ($b_\omega\tau \gg 1$). Thus we have

$$D_\omega = \left(\frac{e}{m}\right)^2 \frac{\delta E_0^2}{2b_\omega^2} \sum_{m=\pm1} \frac{i}{\omega - m\omega_0 + ik^2 D_{n\omega_0-\omega}}. \tag{21}$$

If the turbulence saturation is obtained due to the diffusive compensation of the linear growth rate $\gamma_{\mathbf{k}}$ (that is the case for gradient-driven instability) Eq. (16) yields an estimate of the static value of the diffusion coefficient,
$D_0 = \frac{\gamma_k^3/k^2}{\omega_0^2 + \gamma_k^2}$.
which is in agreement with the observations and mode coupling simulations [6]. This estimate could not be obtained within the Markovian approximation.

References

[1] H. Xia, O. Ishihara, A. Hirose: Phys. Fluids **B 5** 2892 (1993)

[2] R. Balescu: Phys. Rev. **E 51** 4807 (1995)

[3] A. Zagorodny, J. Weiland, A. Jarmen: Comments Plasma Phys. Control. Fusion **17** 353 (1997)

[4] T.G. Mason, D.A. Weitz: Phys. Rev. Lett. **74** 1250 (1995)

[5] F. Amiblard et al.: Phys. Rev. Lett. **77** 4470 (1996)

[6] H. Nordman, J. Weiland: Nuclear Fusion **29** 251 (1989)