NON-BOHM DIFFUSION SCALING DUE TO TRAPPING

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Test particle evolution in turbulent magnetically confined plasmas strongly depends on the Kubo number $K \equiv V\tau_c/\lambda$ that represents the ratio between the correlation time τ_c of the stochastic field and the average transit time $\tau_t = \lambda/V$ of the particles over the correlation lengths λ. Here V is the amplitude of the fluctuating drift velocity determined by the stochastic field $\widetilde{\mathbf{E}} = -\nabla\phi$ perpendicular to the confining magnetic field $\mathbf{B}$ and $\phi(\mathbf{x}, t)$ is the corresponding stochastic potential. A dimensional analysis shows that the general expression of the diffusion coefficient is $D = d(\lambda^2/\tau_c)K^\gamma$ where γ and d are approximately constant in large domains of values of K. In the quasilinear regime $K \ll 1$, the results are well established: the exponent in the diffusion coefficient is $\gamma_{ql} = 2$ and $d = 1$. In the nonlinear regime $K > 1$, all theoretical models lead to a Bohm-like diffusion coefficient with $\gamma_B = 1$. This is not a correct result since it does not vanish in the limit of frozen turbulence $(\tau_c \to \infty)$ as it should. There is only one qualitative estimate [1], based on an analogy with percolation in stochastic landscapes, which determines a subunitary exponent, namely $\gamma_I = 0.7$. The direct numerical calculations of particle trajectories confirm this result [2]. We show that the physical reason of the discrepancy is *particle trapping in the stochastic potential*. The numerically calculated particle trajectories become very complicated in the nonlinear regime: they have some parts trapped in small regions for long time intervals, and other parts which represent fast jumps. Precisely, the Corrsin approximation used in the theoretical approaches to derive a functional dependence of the Lagrangian correlations on the Eulerian correlations of the stochastic velocity field is not able to account for particle trapping. We have developed a new statistical approach which extends beyond Corrsin approximation and describes the influence of particle trapping on the diffusion coefficient. The main ingredient of the model is the concept of *decorrelation trajectory* which determines the dynamics of the decorrelation process.

We consider a 2-dimensional nonlinear Langevin equation:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t), t), \qquad \mathbf{x}(0) = \mathbf{0} \tag{1}$$

where the velocity field $\mathbf{v}(\mathbf{x}, t) = -\nabla\phi \times \mathbf{B}/B^2$ is the drift velocity of the guiding centers of the charged particles in the space $(\mathbf{x})$ and time (t) dependent stochastic potential $\phi(\mathbf{x}, t)$ which is considered to be Gaussian, stationary, homogeneous and isotropic, with zero average. The *Eulerian two-point correlation function* (EC) of the potential is given. This is a measurable quantity defined as the statistical average of the potential in two points $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_2)$. We chose the following model for the EC:

$$E(\mathbf{x}, t) \equiv \langle \phi(\mathbf{x}_1, t_1)\, \phi(\mathbf{x}_2, t_2) \rangle = \beta^2 \exp\left(-|t|\right) / \left(1 + r^2/2n\right)^n . \tag{2}$$

Due to the stationarity and homogeneity conditions this average depends only on $r = |\mathbf{x}_1 - \mathbf{x}_2|$ and on the time interval $t = |t_1 - t_2|$. The distance r is measured in units of λ and the time in τ_c. The symbol $\langle ... \rangle$ denotes the statistical average over the realizations of the stochastic potential and $\beta = \lambda V$ is the amplitude of the potential fluctuations.

The essential point of the new method is that it yields a set of deterministic trajectories which are obtained in terms of the EC; the LC of the velocity is then approximated using the average velocity on these trajectories. The idea is to divide the space of realizations of the stochastic potential into subensembles characterized by fixed values of the potential and of the velocity at the starting point of the trajectories:

$$\phi(\mathbf{0}, 0) = \phi^0, \quad \mathbf{v}(\mathbf{0}, 0) = \mathbf{v}^0. \tag{3}$$

The EC of the velocity components $E_{ij}(\mathbf{x}, t)$ can be decomposed as:

$$E_{ij}(\mathbf{x}, t) = \int \int d\phi^0 \, d\mathbf{v}^0 \, P_1(\phi^0) \, P_1(\mathbf{v}^0) \, E^s_{ij}(\mathbf{x}, t) \tag{4}$$

where $P_1(\phi^0)$ and $P_1(\mathbf{v}^0)$ are the Gaussian probability densities for the initial potential and for the initial velocity. $E^s_{ij}(\mathbf{x}, t) \equiv \langle v_i(\mathbf{0}, 0) \, v_j(\mathbf{x}, t) \rangle \, |_{\phi^0, \mathbf{v}^0} = v^0_i \; \langle v_j(\mathbf{x}, t) \rangle \, |_{\phi^0, \mathbf{v}^0}$ is the subensemble Eulerian correlation, i.e. it is an average conditioned by (3) and $\langle v_j(\mathbf{x}, t) \rangle \, |_{\phi^0, \mathbf{v}^0}$ is the Eulerian average velocity in the subensemble (3). The latter is determined using the Gaussian conditional probability density for having the velocity $\mathbf{v}$ in the point $(\mathbf{x}, t)$ when the condition (3) is imposed. Straightforward calculations lead to:

$$\langle \mathbf{v}(\mathbf{x}, t) \rangle \, |_{\phi^0, \mathbf{v}^0} = \mathbf{f}(\mathbf{x}; \phi^0, \mathbf{v}^0) \, \exp{(-t)} \tag{5}$$

where the factor $\mathbf{f}(\mathbf{x}; \phi^0, \mathbf{v}^0)$ depends on the EC of the potential (2) and on the subensemble. This equation exhibits the space-time structure of the correlated zone. The average velocity in the subensemble (3) is $\mathbf{v}^0$ in $\mathbf{x} = \mathbf{0}$ and $t = 0$ and it decays progressively to zero as the time and/or the distance grows. Both time (through the factor $\exp(-t)$) and distance (through the factor $\mathbf{f}(\mathbf{x}; \phi^0, \mathbf{v}^0)$) determines the decorrelation of the velocity. The space-dependent factor $\mathbf{f}$ describes the structure of the correlated zone. We determine the dynamics induced by this structure by solving the equation:

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{f}(\mathbf{X}(t); \phi^0, \mathbf{v}^0), \qquad \mathbf{X}(0) = \mathbf{0}. \tag{6}$$

The solution of this equation $\mathbf{X}(t; \, \phi^0, \mathbf{v}^0)$ is called *the space decorrelation trajectory* and determines the typical evolution in the correlated zone and the way to leave it. We note that $\mathbf{X}(t; \, \phi^0, \mathbf{v}^0)$ is not an approximation of the average particle trajectory in the subensemble: rather it is a deterministic trajectory which represents the dynamics of the space decorrelation. The decorrelation paths (6) are presented in Fig. 1.a for several subensembles labeled by the values of $p \equiv \phi^0 / \, |\mathbf{v}^0|$. All decorrelation paths are closed curves except the path for $p = 0$ which is the straight line along $\mathbf{v}^0$.

The basic approximation of our model consists in considering that the Lagrangian correlation of the velocity components is a weighted sum of the correlations observed *along the decorrelation trajectories* in each subensemble (3):

$$L_{ij}(t) \cong \int \int d\phi^0 \, d\mathbf{v}^0 \, P_1(\phi^0) \, P_1(\mathbf{v}^0) \, v^0_i \, \left\langle v_j \left(\mathbf{X}(t; \, \phi^0, \mathbf{v}^0), t \right) \right\rangle |_{\phi^0, \mathbf{v}^0} \tag{7}$$

The validity of this approximation will be proved *a posteriori* by the results obtained by our model for several quantities. After straightforward calculations, one obtains $L_{xy}(t) \cong 0$ and

$$L_{xx}(t) \cong L_{yy}(t) \cong \left(\frac{\beta}{\lambda}\right)^2 G(Kt)\,\exp(-t) \tag{8}$$

$$G(Kt) = \frac{1}{\sqrt{2\pi}} \int\int_0^\infty dpdu\; u^4 \exp\left(-\frac{u^2(1+p^2)}{2}\right) \frac{dX(\tau,p)}{d\tau}. \tag{9}$$

Here, $X(\tau, p)$ is the component of the decorrelation trajectories along $\mathbf{v}^0$ and $\tau \equiv Kut$. The diffusion coefficient is determined as the time integral of the LC: $D(K) = \int_0^\infty L_{xx}(t)\,dt$.

Two time factors compete in determining the shape of the Lagrangian correlation (8): the exponential which accounts for the explicit time decorrelation and the function $G(Kt)$ which is determined by the Lagrangian nonlinearity. This function is calculated numerically and is presented in Fig. 1.b.: $G(\theta)$ has a positive part followed by a negative minimum and by a very long negative tail. The positive and negative parts have equal areas so that the integral of $G(\theta)$ is zero: $\int_0^\infty G(\theta)\,d\theta = 0$. At small Kubo numbers the exponential factor prevails. The decorrelation is temporal and $L_{xx}(t) \cong (\beta/\lambda)^2 \exp(-t)$. The nonlinear factor $G(Kt)$ becomes decisive at high Kubo numbers where it provides a time variation faster than that of the exponential factor. The Lagrangian correlation $L_{xx}(t)$ at $K \gg 1$ has a shape similar to $G(Kt)$ but with the negative tail more attenuated because of the exponential factor, Fig. 2. It is very similar with the LC of the velocity determined in the direct numerical simulations [2].

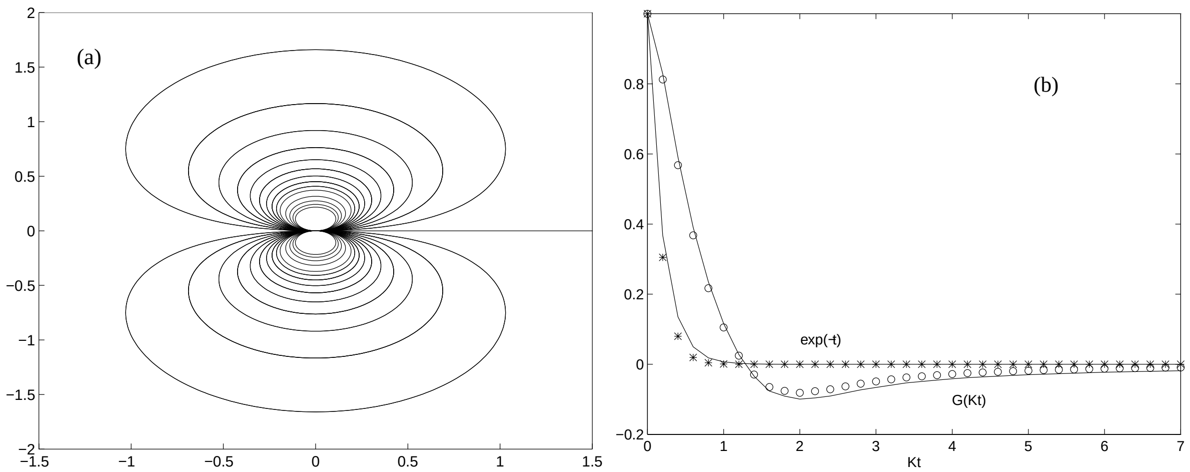

Figure 1. (a) The decorrelation paths for $p = 0, \pm 0.5, \pm 1, ...$ The size of the curves decreases continuously with p; **(b)** The LC for $K = 0.2$ (stars) and 10 (circles), as a function of Kt, compared to the two factors in Eq. (8). At small K the LC is close to $\exp(-t)$ while at large K it has the shape of $G(Kt)$.

The large K numerical simulations of particle trajectories show that during their evolution the particles are temporarily trapped on almost closed, small size paths for durations long enough for performing a large number of rotations. Such trapping events appear around the extrema of the potential while long displacements are performed when the particles are at small values of the potential. Our model gives an image of this rather complicated trapping process which is actually contained in the Lagrangian correlation, Eq. (8). The shape of the nonlinear factor $G(\theta)$ is determined by a selected contribution of the various paths (i.e. subensembles). The small paths with large value of $|p|$ (i.e. of the potential ϕ^0) contribute only at the peak of $G(\theta)$

at $\theta = 0$. At later times, since these trajectories perform a large number of rotations, their contributions cancel by an incoherent mixing in the integral over u. The negative tail of $G(\theta)$ results only from the contributions of the large paths corresponding to $|p| \ll 1$. When there is no time variation of the stochastic potential ($\tau_c \to \infty$) the asymptotic diffusion coefficient is zero: $D \sim \int_0^\infty G(\theta)\, d\theta = 0$ and the process is subdiffusive. A slow time variation ($\tau_c \gg 1$ or $K \gg 1$) produces the attenuation of the negative tail of $G(\theta)$, thus the elimination of the large paths contribution to the Lagrangian correlation, in other words the decorrelation of those trajectories. A nonzero diffusion coefficient results from this escape of the large size trajectories. Actually, the diffusion is produced only by the latter trajectories and not by the small ones whose contribution is not affected by the time decorrelation. When the time variation becomes fast ($\tau_c \ll 1$ or $K \ll 1$), all trajectories are decorrelated and the function G does not influence the diffusion coefficient. Thus, the two factors in the Lagrangian correlation, Eq. (8), have a clear physical interpretation: $G(Kt)$ describes the trapping of particles on the equipotential lines while the linear factor $\exp(-t)$ accounts for the trajectory release. The Lagrangian correlation and consequently the diffusion coefficient results from the competition between trapping and release processes.

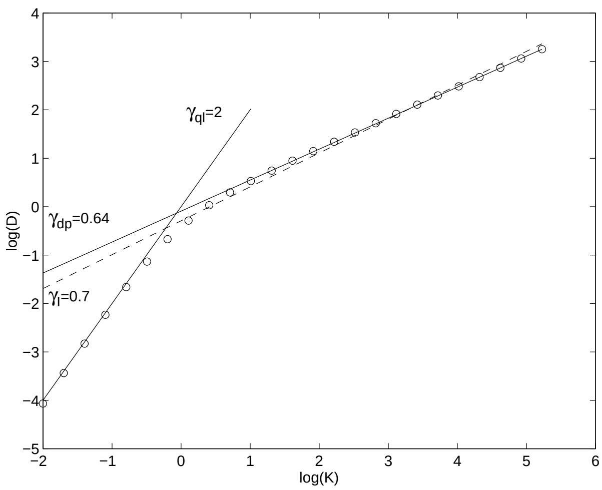

Figure 2. The diffusion coefficient obtained with the decorrelation path method (circles) compared to the quasilinear and percolation scalings.

In Fig. 2 we present the K dependence of the asymptotic diffusion coefficient obtained from the Eulerian correlation (2) with $n = 0.85$, a value measured in [2]. After the small K quasilinear regime appearing for $K \leq 1$, a slower dependence on Kubo number is observed. The K dependence of the diffusion coefficient is weaker than in the Bohm scaling. The diffusion coefficient can be approximated as $D \sim K^{0.64}$. As seen in Fig. 2, the results of our model are close to the percolation scaling [1]. The exponent of K is proved to depend slowly on the parameter n describing the decay of the EC.

Conclusions

We have presented an analytical approach for particle diffusion in 2-dimensional incompressible stochastic velocity fields which is able to describe the complex process of diffusion and intrinsic trapping in plasma turbulence. Particle trapping determines a reduction of the diffusion coefficient and consequently a change of the scaling in K which is no more of Bohm type.

References

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