

FORCE-FREE MHD EQUILIBRIA WITH NON-CONSTANT $\lambda(\Phi)$

J.W. Edenstrasser and F. Mairey

Institute for Theoretical Physics at the University of Innsbruck, Austria

1. Introduction

Based on a multiple timescale derivative expansion scheme, J.W. Edenstrasser [1] has in the leading zeroth-order arrived at the general force-free MHD equilibria $\text{curl } \vec{B}_0 = \lambda(\Phi_0) \vec{B}_0$, where \vec{B}_0 is the zeroth-order magnetic field and Φ_0 the corresponding flux function. In the context of his famous relaxation theory, J.B. Taylor [2] has obtained the special force-free equilibria with constant λ . Taylor's theory became very popular, particularly in the context of the reversed field pinches (RFPs), where it predicted the reversal of the toroidal magnetic field at the outer part of the discharge. At these special force-free equilibria he has arrived by minimizing the total magnetic energy within the plasma and posing the overall helicity $K := \int \vec{A} \cdot \vec{B} d\tau$ as a variational constraint.

On the ideal MHD (Alfvén) timescale the local K^ϕ is an invariant of motion for each single flux tube. Taylor's great merit now consists in the conjecture, that for a "slightly imperfect" plasma (corresponding to the evolution on an intermediate timescale) only the overall K remains as an invariant of motion. Taylor's conjecture has been proven by J.W. Edenstrasser and M.M. Kassab [3], where it is shown that on the intermediate MHD collision timescale of a fusion plasma indeed only the overall K remains as an invariant of motion.

In spite of the striking success of Taylor's relaxation theory concerning the prediction of the field reversal of RFPs and the qualitatively rather good agreement between the theoretically predicted and experimentally measured so-called F- Θ -diagrams, there remained many open questions for which Taylor's theory could not provide a sufficient explanation. Thus, in analyzing RFP equilibria in the literature a non-constant $\lambda(\phi)$ was considered, where in the case of the radially symmetric, cylindrical equilibria $\lambda(\phi)$ was expanded in a power series with respect to the radial coordinate r [4]. The coefficients a_n in the expansion $\lambda(r) = \sum a_n r^n$ have then been determined empirically from the magnetic field and current density profile measurements. This fact has motivated us to determine the flux function $\lambda(\Phi_0)$, appearing in the zeroth-order equilibrium equation by some minimum energy principle.

2. The symmetric force-free equilibria

Consider the case of symmetric equilibria (i.e. to the case of an ignorable coordinate x^3). Symmetry requirement $\partial/\partial x^3 = 0, \nabla B_0 = 0$

$$\Rightarrow \vec{B}_0 = \vec{g}^3 \times \nabla \Phi_0 + \frac{1}{g_{33}} B_{3(0)}(\Phi_0) \vec{g}_3, \quad (1)$$

Φ_0 is the zeroth-order poloidal magnetic flux per unit of the symmetry variable x^3 and measured from the magnetic axis outwards.

In a general, non-orthogonal curvilinear coordinate system $\{x^1, x^2, x^3\}$ satisfying $\partial g_{ik}/\partial x^3 = 0$, the contravariant components of the zeroth-order magnetic field are given by

$$B_{(0)}^1 = -\frac{1}{\sqrt{g}} \frac{\partial \Phi_0}{\partial x^2}, \quad B_{(0)}^2 = \frac{1}{\sqrt{g}} \frac{\partial \Phi_0}{\partial x^1}, \quad B_{(0)}^3 = g^{31} B_1 + g^{32} B_2 + g^{33} B_{3(0)}(\Phi_0) \quad (2)$$

implying in the zeroth-order force-free case for the current density the representation

$$\vec{j}_0 = \text{curl } \vec{B}_0 = -\frac{dB_{3(0)}}{d\Phi_0} \vec{B}_{(0)} \stackrel{!}{=} \lambda(\Phi_0) \vec{B}_{(0)} \quad \Rightarrow \quad \boxed{\lambda(\Phi_0) = -\frac{dB_{3(0)}}{d\Phi_0}}. \quad (3)$$

For a given functional dependence $B_{3(0)}(\Phi_0)$ the generalized Grad-Shafranov equation, which reduces in this constant pressure force-free case to the form,

$$\nabla \left(\frac{\nabla \Phi_0}{g_{33}} \right) + B_{3(0)} \nabla \cdot \vec{a} + \frac{1}{2g_{33}} \frac{dB_{3(0)}^2(\Phi_0)}{d\Phi_0} = 0, \quad \vec{a} := \left(\frac{\vec{g}_3 \times \vec{g}^3}{g_{33}} \right). \quad (4)$$

can be solved for the poloidal flux function Φ . In order to determine the functional dependence $\lambda(\Phi_0)$ the total magnetic energy

$$W_0 = \int \frac{\vec{B}_0^2}{2} d\tau = \int \frac{|\nabla \Phi_0|^2 + B_{3(0)}^2(\Phi_0)}{2g_{33}} d\tau \quad (5)$$

is minimized by keeping a few overall physical quantities constant. Here, our basic assumption is that in the stationary phase of the discharge (flat top) the minimum energy state is not determined by relaxation constraints on shorter timescales (like K conservation by J.B.Taylor [2]), but rather by the overall fluxes, which are in the hand of the experimenter.

A. The case $\mathbf{J}_{tor} = \Phi_{pol} = const$:

In this case the magnetic energy is minimized $W_0 \rightarrow Min$ by taking the total toroidal current \mathbf{J}_{tor} and total poloidal magnetic flux Φ_{pol} as variational constraints leading to the solution

$$\boxed{B_{3(0)}(\Phi_0) = \left\{ \left| a + b e^{-\alpha \Phi_0} \right| \right\}^{\frac{1}{2}}}, \quad (6)$$

where the constants a,b,and α are determined by boundary conditions.

B. The case $\mathbf{J}_{tor} = \Phi_{pol} = \Psi_{tor} = const$:

In this case in addition also the total toroidal magnetic flux Ψ_{tor} is taken as variational constraint leading to the solution

$$\boxed{B_{3(0)}(\Phi_0) = \left\{ c + d e^{-\gamma \Phi_0} \right\}^1}, \quad (7)$$

where the constants c,d, γ are determined either by the constraints $\mathbf{J}_{tor}, \Phi_{pol}, \Psi_{tor}$ or by some other boundary conditions.

3. The radially symmetric, cylindrical solutions

If one applies for the normalization of the magnetic field the value of the toroidal magnetic field at the axis and for the poloidal flux the value at the wall, i.e. $\tilde{B}_z = \bar{B}_z(0) B_z(\Phi)$, $\tilde{\Phi} = \bar{\Phi}_{wall} \Phi(x)$, and $r = r_0 x$, then the general equilibrium equation (4) reduces for the radially symmetric cylindrical case to the ordinary differential equation

$$\frac{d^2\Phi}{dx^2} + \frac{1}{x} \frac{d\Phi}{dx} + \kappa^2 B_z \frac{dB_z}{d\Phi} = 0, \quad \kappa := \frac{\bar{B}_z r_0}{\bar{\Phi}_{wall}} = -\frac{2}{B'(0) q_{ax}/\varepsilon}, \quad (8)$$

together with the boundary conditions $\Phi(0) = 0$ and $\Phi(1) = 1$ (q_{ax} is the safety factor on axis and ε the inverse aspect ratio). Furthermore, with this normalization the poloidal and toroidal components of the magnetic field and of the current density read

$$B_\varphi = \frac{1}{\kappa} \frac{d\Phi}{dx}, \quad B_z = B_z(\Phi); \quad j_\varphi = -\frac{d\Phi}{dx} \frac{dB_z}{d\Phi}, \quad j_z = -\kappa \frac{dB_z}{d\Phi} B_z(\Phi). \quad (9)$$

A. The case $\mathbf{J}_{tor} = \Phi_{pol} = const$:

For this case $B_z(\Phi)$ is given by Eq. (6), leading to $B_z(\Phi) \frac{dB_z}{d\Phi} = -\frac{1}{2} \alpha b e^{-\alpha\Phi}$, so that the resulting equilibrium equation (8) possesses the analytical solution [6]

$$\Phi = \frac{\ln(1 + \mu x^2)}{\ln(1 + \mu)}, \quad \mu = \frac{b\alpha^2 \kappa^2}{16} = \frac{1}{b\rho^2} \stackrel{!}{=} e^{\frac{\alpha}{2}} - 1; \quad \rho = q_{ax}/\varepsilon, \quad (10)$$

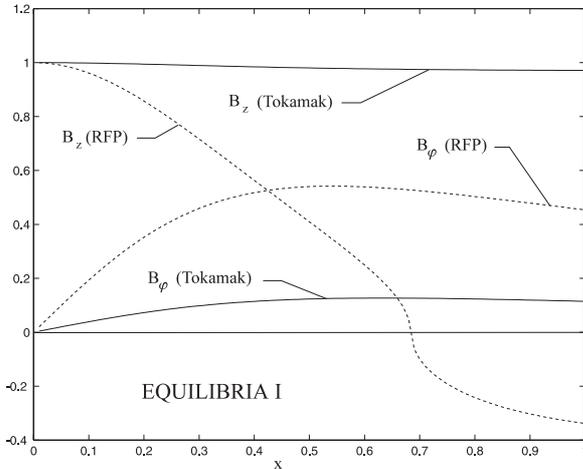


Figure 1.

$$B_\varphi = \frac{1}{\rho} \frac{x}{(1 + \mu x^2)}, \quad \mu = \frac{1}{\rho^2 b}, \quad B_z = \pm \sqrt{\left| \pm 1 - b + \frac{b}{(1 + \mu x^2)^2} \right|},$$

$$j_\varphi = \frac{2}{\rho^2} \frac{x}{(1 + \mu x^2)^3} \left| \pm 1 - b + \frac{b}{(1 + \mu x^2)^2} \right|^{-\frac{1}{2}}, \quad j_z = \frac{1}{\rho} \frac{1}{(1 + \mu x^2)^2}. \quad (11)$$

In Fig. 1, there are represented the magnetic field profiles for the paramagnetic tokamak as well as for the RFPs, in both cases showing an at least qualitatively good agreement with the experimental ones. In the case of the tokamaks also the current density profiles behave as expected, strongly decreasing towards the plasma edge. The profiles of the diamagnetic tokamak differ from those of the paramagnetic one essentially only in the outwardly increasing toroidal

where the constant b has to be determined by some appropriate boundary conditions. It turns out that this solution not only allows the description of para- and diamagnetic tokamak equilibria but also of RFP's, where the cases $\alpha < 0$ correspond to a hollow toroidal current density profile. The explicit expressions for the magnetic field and current density profiles are then given by:

magnetic field. Note that the toroidal current density profile in Eq. (11) is identical to the one found by Kadomtsev within his theory of optimized profiles [7]. However, in the case of the RFP's the poloidal current density becomes singular at the reversal radius. This fact emphasizes once more the necessity to include in a comparison of theoretically predicted RFP equilibria with the experimental ones also the current density profiles.

B: The case $\mathbf{J}_{tor} = \Phi_{pol} = \Psi_{tor} = \text{const:}$

After normalizing the toroidal field given by Eq. (7) to unity on the magnetic axis, i.e. $B_z(0)=1$, one obtains for the components of the magnetic field

$$B_\varphi = \frac{1}{\kappa} \frac{d\Phi}{dx}, \quad B_z(\Phi) = c + (1 - c) e^{-\gamma\Phi}, \quad c := \frac{R - e^{-\gamma}}{1 - e^{-\gamma}} \quad (12)$$

leading to the non-linear eigen value problem for γ and $\Phi(x; \gamma)$

$$\frac{d^2\Phi}{dx^2} + \frac{1}{x} \frac{d\Phi}{dx} - \frac{4}{\rho^2\gamma} \left\{ \frac{c}{1-c} e^{-\gamma\Phi} + e^{-2\gamma\Phi} \right\} = 0, \quad \kappa = -\frac{2}{\rho\gamma(1-c)} \quad (13)$$

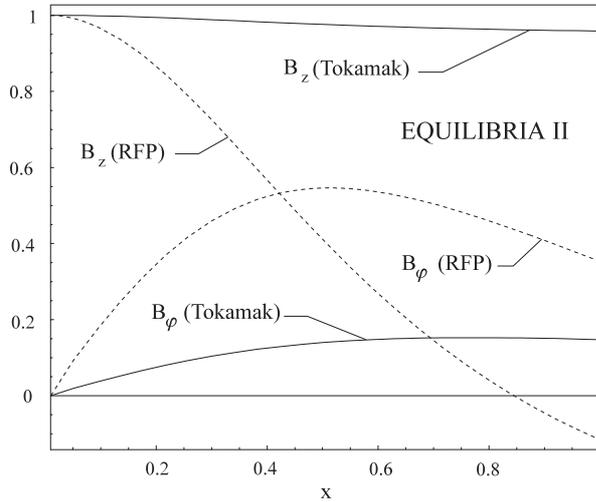


Figure 2.

with the boundary conditions $\Phi(0) = 0$, $\Phi(1) = 1$ and where the constant c is determined by the prescription of the toroidal field at the wall $B_z(1) = R$. Eq. (13) is solved numerically and represented in Fig. 2 for different values of ρ and R . It turns out that these equilibria can equally well describe both the para- and diamagnetic tokamaks as well as RFPs in an at least qualitatively good agreement with experimental observations. Concerning a quantitative agreement and stability behavior further investigations have to be performed.

Acknowledgements

This work was supported in part by the Austrian ÖAW-Association agreement, Project P8.

References

- [1] Edenstrasser, J.W.: Phys. Plasmas **2** (4), 1192 (1995)
- [2] Taylor, J.B.: Phys. Rev. Lett. **33**, 1139 (1974)
- [3] Edenstrasser, J.W. and Kassab, M.M.: Phys. Plasmas **2**(4), 1206 (1995)
- [4] Ortolani, S. & Schnack, D.D.: Magnetohydrodynamics of Plasma Relaxation. World Scientific Publishing Co. Pte. Ltd., Singapore (1993), Ch. (4)
- [5] Edenstrasser, J.W.: J. Plasma Phys. **24**, 299 (1980)
- [6] Edenstrasser J.W.: "Stable High- β Equilibria for Cylindric Symmetry." *Proc. of the Varenna Workshop on High- β Equilibria* (1977). Publ. ERDA-Washington, p.17
- [7] Kadomtsev, B.B.: Sov. J. Plasma Phys. **13**(7), 443 (1987)