STATISTICS OF 2-D VORTICES AND HOLTSMARK'S DISTRIBUTION

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In this paper, a new approach to the problem of point vortices statistics is proposed. It's based on the consideration of non–correlated ensemble (or "gas") of vortices. Such ergodic situation gives an opportunity to obtain some universal features of vortices statistical behavior despite of their different physical nature.

As it is well known, 2-d point vortices on xy plane are characterized by their "charge" (intensity) a_j and "potential" (stream function) ψ (different for different physical situation), which are responsible for the energy of system

$$H = \sum_{j>k} a_j a_k \psi(|\mathbf{r}_j - \mathbf{r}_k|) \tag{1}$$

(canonical variables for this Hamiltonian are $a_j x_j$ and $a_j y_j$) and the motion of each vortex in it

$$\mathbf{v}_j \equiv \dot{\mathbf{r}}_j = \mathbf{e}_z \times \nabla \sum_{k \neq j} a_k \psi(|\mathbf{r}_j - \mathbf{r}_k|). \tag{2}$$

It's seen, that for these particles configuration space is identical to phase one.

Here for the sake of simplicity only case with equal charges $a_j = a_0 =$ const (the reason of it may be quantum effects) and power–like potentials $\psi = g r^{-\alpha}$ ($\alpha \geq 0$, g is physical dimensional coefficient) is investigated. According (2) the velocity of each vortex j is defined by sum

$$\sum_{k \neq j} \nabla \psi(|\mathbf{r}_j - \mathbf{r}_k|),$$

and consequently its statistics (analog of Maxwell distribution function for usual Newtonian particles) is directly associated with the statistics of electric microfields in plasmas [1] (with some corrections for the dimension of problem and type of potential).

It is believed, that if the number of vortices in system N is sufficiently large, their positions on xy plane in any time are random and non–correlated (it's analog of common "ergodic hypothesis"). In this case desired velocity distribution is the same as Holtsmark's one: the probability $f(\mathbf{v})\, d^2\mathbf{v}$ to find a vortex with a speed between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ is equal to the fraction of volume (square) of configuration (phase) space $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N-1}$ of other vortices, where the equality

$$\mathbf{v} = a_0 \sum_{j=1}^{N-1} \nabla \psi(|\mathbf{r}_j|)$$

is realized (a rotation on the angle $\pi/2$ doesn't important here, the final expression for f depends from $|\mathbf{v}|$ only), that is

$$f(\mathbf{v}) = \langle \delta(\mathbf{v} - a_0 \sum_j \nabla\psi(r_j)) \rangle = \frac{1}{(2\pi)^2} \int \exp(\imath \mathbf{q}\mathbf{v}) \langle \exp(-\imath a_0 \mathbf{q} \sum_j \nabla\psi(r_j)) \rangle \, d^2\mathbf{q}, \qquad (3)$$

where angle brackets mean the averaging over this space of non–correlated vortices (i.e. over $\prod_{j=1}^{N-1} d^2\mathbf{r}_j/S$, S is the square of system, see [1]). Further calculations are done for two cases, what give usual type of Holtsmark's distribution and a significantly different from it.

If $\alpha = 1$ (Hall vortices in thin layer of plasma, or Pearl's vortices in superconducting film with distances between them much larger than $l = c^2/(\omega_{pe}\delta)$, where δ is a thickness of layer–film), then after standard procedure in the limit of $N, S \to \infty,\ N/S \equiv n = \mathrm{const}$ (n is a mean vortices density)

$$\langle \ldots \rangle = \int \cdots \int \exp\left(\imath a_0 g \mathbf{q} \sum_{j=1}^{N-1} \frac{\mathbf{r}_j}{r_j^3} \right) \prod_{j=1}^{N-1} \frac{d^2\mathbf{r}_j}{S} = \left[\int \exp\left(\imath a_0 g \frac{\mathbf{q}\mathbf{r}}{r^3} \right) \frac{d^2\mathbf{r}}{S} \right]^{N-1} \equiv$$

$$\equiv \left\{ 1 - \int \left[1 - \exp\left(\imath a_0 g \frac{\mathbf{q}\mathbf{r}}{r^3} \right) \right] \frac{d^2\mathbf{r}}{S} \right\}^{N-1} \to$$

$$\to \exp\left\{ -n \int \left[1 - \exp\left(\imath a_0 g \frac{\mathbf{q}\mathbf{r}}{r^3} \right) \right] \, d^2\mathbf{r} \right\} = \exp(-n\pi |a_0| g q)$$

we have

$$f(\mathbf{v}) d^2\mathbf{v} = f(v) \cdot 2\pi v dv = \frac{n\pi |a_0| g}{\left[(n\pi a_0 g)^2 + v^2 \right]^{3/2}} \, v dv. \qquad (4)$$

As for usual situation the asymptotic behavior of f when $v \to \infty$ is defined by its nearest neighbor (the probability to find it on the small distance is $n \cdot 2\pi r dr|_{r\to 0} \equiv 2\pi f(v) v dv|_{v\to\infty}$ in terms of that $v = a_0 g/r^2$).

Function (4) is automatically normalized to one, but have all divergent moments (even $\langle v \rangle$). They become finite after taking into consideration non–point structure of vortices or more slower, than $1/r$, increasing of ψ when $r \le l$.

If $\alpha = 0$ ($\psi = -\ln r$, the case of ideal liquid or layer vortices with distances $r \ll l$), then there is logarithmic divergence of integral in

$$\exp\left\{ -n \int \left[1 - \exp\left(-\imath a_0 g \frac{\mathbf{q}\mathbf{r}}{r^2} \right) \right] \, d^2\mathbf{r} \right\}$$

for $r \to \infty$. This situation is corrected by change of asimptotics at $r \sim l$ too. After cut–off procedure in this integral at large distance we have in (3)

$$\langle \ldots \rangle = \exp\left[-\frac{\pi}{2} \Lambda n (a_0 g q)^2 \right],$$

where $\Lambda \sim \ln n l^2$ is logarithmic large factor (of course, in fact Λ is a function of q, roughly speaking $\Lambda \sim 1/2 \cdot \ln[1 + 2l/(|a_0| g q)]$, but here it's not so important), therefore

$$2\pi f(v) v dv = \frac{v dv}{\pi \Lambda n (a_0 g)^2} \, \exp\left(-\frac{v^2}{2\pi \Lambda n (a_0 g)^2} \right). \qquad (5)$$

This formula is quite equivalent to the expression which was obtained in [3] as a result of simulations of 2-d point vortices dynamics in ideal liquid (there cut–off distance l was a distance of system space periodicity).

(5) is extremely different from classical Holtsmark's distribution (but very usual from the other point of view): its asymptotics isn't defined by nearest neighbor. The divergence at large r manifests itself indirectly at small r. Indeed, let's suppose that given vortex is situated on the boundary of circle with radius R. There are $\sim \pi n R^2$ vortices inside of circle and statistic fluctuation of their number is $\sqrt{\pi n R^2} \propto R$ (a mean field $\nabla\psi$ of these vortices is compensated by others on the other side of the given one, namely fluctuations dictate this situation), hence their contribution into velocity $R/R = \mathrm{const}$, i.e. roles of large and small scales are identical. Most likely, that is a reason of Gauss–like form of f.

The one–particle energy distribution function for ergodic vortices ensemble can be obtained in the same manner (see (1)):

$$f(\varepsilon) = \langle \delta(\varepsilon - a_0^2 \sum_j \psi(r_j)) \rangle = \frac{1}{2\pi} \int \exp(\imath\omega\varepsilon) \langle \exp(-\imath a_0^2 \sum_j \psi(r_j)) \rangle \, d\omega. \qquad (6)$$

Here the divergence exists for $\alpha \leq 2$. Let's consider two cases again.

For $\alpha = 4$ (van der Waals 2-d interaction of 3-d vortices on large distances in layered superconductors along these layers [4])

$$f(\varepsilon)d\varepsilon = \theta(\varepsilon) \frac{\pi a_0 g^{1/2}}{2\varepsilon^{3/2}} \exp\left(-\frac{\pi^3 n^2 a_0^2 g}{\varepsilon} \right) d\varepsilon. \qquad (7)$$

The behavior when $\varepsilon \to \infty$ is defined by nearest neighbor and all moments diverge again.

For $\alpha = 1$ it's necessary to cut–off intermediate integral at some L (it may be the distance of periodicity or the size of a system $nL^2 \sim N$) and

$$f(\varepsilon) = \delta(\varepsilon - 2\pi n a_0^2 g L). \qquad (8)$$

Now we can compare energies of gas and lattice of vortices. The former is always larger, but difference depends on absence or presence of divergence on r. In first variant regular lattice forbids the unlimited rapprochement and the energy of given vortex remains finite, whereas, for example, (7) gives $\langle \varepsilon \rangle = \infty$. The change of ψ at $r \to 0$ corrects situation, but in any case energy of chaotic state is much greater than lattice one. In the second, since the digitization on large distance doesn't so appreciable, the difference on the contrary small (of the order of $N^{(\alpha-2)/2}$).

Of course, energy excess makes more difficult the realization of ergodic state. If system can "drop" additional energy, gas condensates into crystal lattice. But in the opposite case non–correlated state is possible and interesting too. The more so that in most real situations $\alpha < 2$ and energy discrepancy is small.

So, we see that ergodic ensemble of vortices can actually be described by Holtsmark's approach for different physical situations (different ψ). All these situations have one common feature — n is a single parameter of distribution functions including the measure of their width. One can say that for gas of 2-d vortices their density is a temperature.

References

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