TWO-POTENTIAL DIPOLE VORTICES

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1. Vortices in plasmas

Electromagnetic vortices are considered to play an important role in the spatial self-organization of a plasma. They are thought to be one of the possible sources of turbulence and of enhanced particle and energy transport. Nonlinear electrostatic perturbations in a plasma lead to so-called one-potential drift vortices [1]. Electromagnetic vortices, however, depend on two scalar functions:the electrostatic potential and one component of the vector potential [2]. Although many particular types of two-potential vortices have been considered in the literature, their general dispersion properties have not been analyzed in detail. In this paper we present the results of such an analysis.

2. Equations for two-potential vortices

Two-potential vortices are described by the system of coupledequations

$$\nabla_\perp^2 \Phi_1 = w_{11}\Phi_1 + w_{12}\Phi_2, \qquad \nabla_\perp^2 \Phi_2 = w_{21}\Phi_1 + w_{22}\Phi_2. \tag{1}$$

Here, the matrix elements w_{ij} are piecewise constants and $\Phi_{1,2}$ are streaming potentials of the flows. As an example, Eqs. (1) are derived in [3] within the framework of the two-fluid model of a low-beta inhomogeneous plasma. These equations govern drift-Alfvén vortices that propagate with a constant velocity perpendicularly to both the magnetic field and the density gradient. In this case the functions $\Phi_{1,2}$ are linear combinations of the electrostatic and vector potentials, with asymptotic behaviour $\Phi_{1,2} \propto -x$ at $|x| \to \infty$. These are typical boundary conditions for vortices propagating in plasmas with a density gradient. From the boundary conditions it follows that at infinity one of the eigenvalues of the matrix w_{ij} is zero.

In what follows we investigate solutions of Eqs. (1) that are finite at $r = 0$, satisfy the prescribed boundary conditions at infinity, and are continuous together with their derivatives up to second order. Such solutions can not be found unless some of the constants w_{ij} have different values inside and outside a separatrix surface. We consider the case of a single circular separatrix located at the radius $r = r_s$. Continuity of the second derivatives requires that one of the functions Φ_i must vanish at the separatrix. Without loss of generality we take $\Phi_1(r_s) = 0$. This implies that the matrix elements w_{i2} do not change their values, while w_{i1} have jumps across the separatrix.

3. Solutions of vortex equations

Equations (1), together with the boundary conditions mentioned above, admit solutions in the form $\Phi_j = \phi_j(r)\cos\theta$, where $(r,\ \theta)$ are polar coordinates. The explicit expressions for the functions $\phi_j(r)$ are

$$\phi_1^{(e)} = -\Big[A\frac{r_s}{r} + \frac{r}{r_s} - (1+A)\frac{K_1(k_e r)}{K_1(k_e r_s)}\Big], \tag{2}$$

$$\phi_2^{(e)} = -\Big[A\frac{r_s}{r} + \frac{r}{r_s} - \frac{w_{22}}{w_{22}-k_e^2}(1+A)\frac{K_1(k_e r)}{K_1(k_e r_s)}\Big], \tag{3}$$

in the region outside the separatrix ($r \geq r_s$), and

$$\phi_1^{(i)} = (1+A)\,\frac{k_e^2}{k_1^2-k_2^2}\,\Big[b(k_1 r_s) - b(k_2 r_s)\Big], \tag{4}$$

$$\phi_2^{(i)} = (1+A)\,\frac{k_e^2}{k_1^2-k_2^2}\,\frac{w_{22}-k_2^2}{w_{22}-k_e^2}\Big[b(k_1 r_s) - \frac{w_{22}-k_1^2}{w_{22}-k_2^2}b(k_2 r_s)\Big]. \tag{5}$$

in the region inside the separatrix ($r \leq r_s$). Here, k_e is the nonzero eigenvalue of w_{ij} outside the separatrix, $k_{1,2}$ are the eigenvalues of w_{ij} inside the separatrix and b are Bessel functions of the first order: $b(kr) = I_1(kr)/I_1(kr_s)$ for $k^2 > 0$,$b(kr) = J_1(|k|r)/J_1(|k|r_s)$ for $k^2 < 0$. Localized vortices require $k_e^2 > 0$. The amplitude A is determined by the equation

$$A = -\frac{k_2^2\mathcal{B}(k_1 r_s) - k_1^2\mathcal{B}(k_2 r_s)}{k_2^2\mathcal{B}(k_1 r_s) - k_1^2\mathcal{B}(k_2 r_s) - 2(k_1^2-k_2^2)}, \tag{6}$$

where $\mathcal{B}(k) \equiv kI_2(k)/I_1(k)$ for real k and $\mathcal{B}(k) \equiv -|k|J_2(|k|)/J_1(|k|)$ for imaginary k. We have normalized ϕ_i such that asymptotically $\phi_i = -x$ at $x \to \infty$.

4. Dispersion relation

The matching conditions across the separatrix yield the dispersion equation that relates the values of k_j in the internal and external regions,

$$(k_1^2-k_2^2)\mathcal{K}(k_e r_s) + (k_e^2-k_2^2)\mathcal{B}(k_1 r_s) - (k_e^2-k_1^2)\mathcal{B}(k_2 r_s) = 0, \tag{7}$$

where $\mathcal{K}(k) \equiv kK_2(k)/K_1(k)$. Equation (7) has been obtained in [3]. In analysing Eq. (7) we consider k_e as given and k_1, k_2 as unknowns. Its derivation as well as the derivation of Eqs (2) – (6)can be found in [4] which contains a detailed analysis of drift-Alfvén vortices with finite ion gyroradius and electron inertia effects.

The dispersion relation (7) possesses the symmetry $k_1^2 \leftrightarrow k_2^2$ and it has no solutions in the quadrant $(k_1^2, k_2^2) > 0$. Figure 1 shows the solutions of Eq. (7) in the region $k_1^2 < (0, k_2^2)$ for three different values of k_e: $k_e r_s = 0$, $k_e r_s = 5$ and $k_e r_s = \infty$. We classify the vortices depending on the number of different roots of Eq. (7) for a fixed k_e and k_1^2. It will be shown that the number of roots is equal to the number of zeros of the function $\phi_1(r)$ inside the separatrix. The function $\phi_1(r)$ depends only on the eigenvalues of the matrix w_{ij}, while $\phi_2(r)$ also depends on the element w_{22}, i.e. on the specific physical model. For this reason we do not analyse the behaviour of the function $\phi_2(r)$. The first two zones, I and II, where Eq. (7) with $k_e = 5$ has one and two roots respectively, are shown in Fig. 1. The zones are bounded by the dashed, vertical lines which are tangent to the asymptotic solutions for $k_2^2 r_s^2 \to \infty$.

In order to find k_1 and k_2 explicitly, the dispersion relation (7) must be supplemented by an additional relationship between k_1 and k_2 which follows from the conditions $k_1^2 + k_2^2 = \mathrm{Sp}(w_{ik})$ and $k_1^2 k_2^2 = \mathrm{Det}(w_{ik})$. The restrictions introduced by this additional relationship are not considered here.

5. Spatial structure of dipole vortices

Moving along a curve of fixed k_e, say $k_e r_s = 5$, starting at a positive value of $k_2^2 r_s^2$, several special points are encountered that are associated with the radial behaviour of the function $\phi_1(r)$. We move along the curve that starts in zone I. The radial profiles, the 2D spatial structures and the levels of the potential $\phi_1(r)$ are given in Figs 2–4. For positive values of $k_2^2 r_s^2$ the function $\phi_1(r)$ vanishes only at $r = 0$ and at the separatrix $r = r_s$. The corresponding potential is given in Figs 2. For negative values of $k_2^2 r_s^2$, $\phi_1(r)$ develops a zero in the outer region $r > r_s$. The birth of this zero occurs at the points where $A \to \infty$. These points are marked by the black dots in Fig. 1. They lie on the intersections of the two dispersion curves $k_e r_s = 5$ and $k_e r_s = 0$. The behaviour of $\phi_1(r)$ just beyond the point mentioned is shown in Figs 3. From these figures it is seen that the additional zero in $\phi_1(r)$ results into a second circular separatrix. However, at this separatrix all the derivatives are continuous in contrast to their behaviour at the $r = r_s$ separatrix. For increasingly negative values of $k_2^2 r_s^2$, this zero finally moves inside the surface $r = r_s$. This occurs at the intersection point of the dispersion curves $k_e r_s = 5$ and $k_e r_s = \infty$ and corresponds to the transition to zone II. The radial structure, the 2D structure and the levels of a vortex in zone II are shown in Figs 4. This pattern of the birth of extra zeros in $\phi_1(r)$ is repeated when one follows a $k_e r_s = constant$ curve towards increasingly negative values of $k_2^2 r_s^2$.

In Fig. 5 we show how the $k_1^2 r_s^2$ coordinates of these special points depend on $k_e r_s$. The solid lines mark the boundaries of the first three zones. The points, where a new zero in $\phi_1(r)$ appears, lie on the long-dashed lines.

6. Conclusions

We have investigated the dispersion properties of two-potential dipole vortices. Our analysis is general, since it does not depend on the particular vortex type, and relies only on Eqs. (1). For the case of potentials which are linear at infinity, we have derived the general dispersion Eq. (7) that relates the eigenvalues of the matrix w_{ij} inside and outside the separatrix. Regularities in the behaviour of the vortex spatial structure are identified and a classification of the vortices is given according to the number of roots of the dispersion relation.

References

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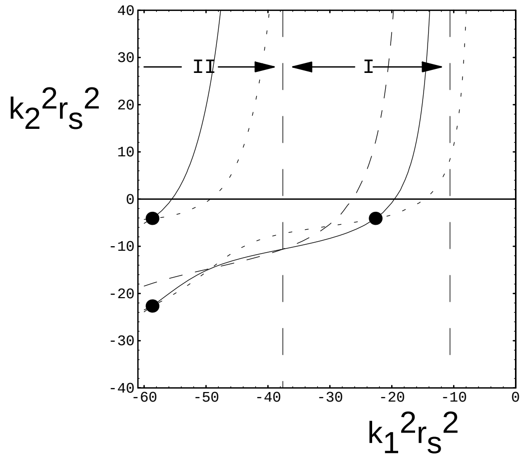


Fig.1 Dispersion curves for $k_e r_s=0$ (dotted), $k_e r_s=5$ (drawn), $k_e r_s=\infty$ (dashed) and zone boundaries (long dashed)

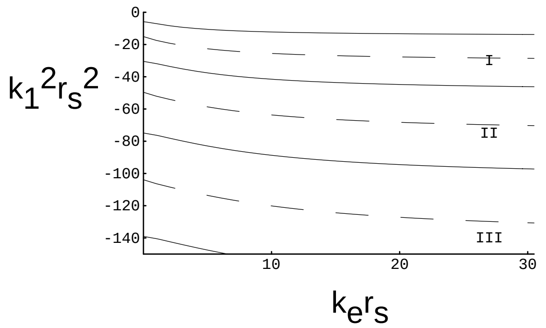
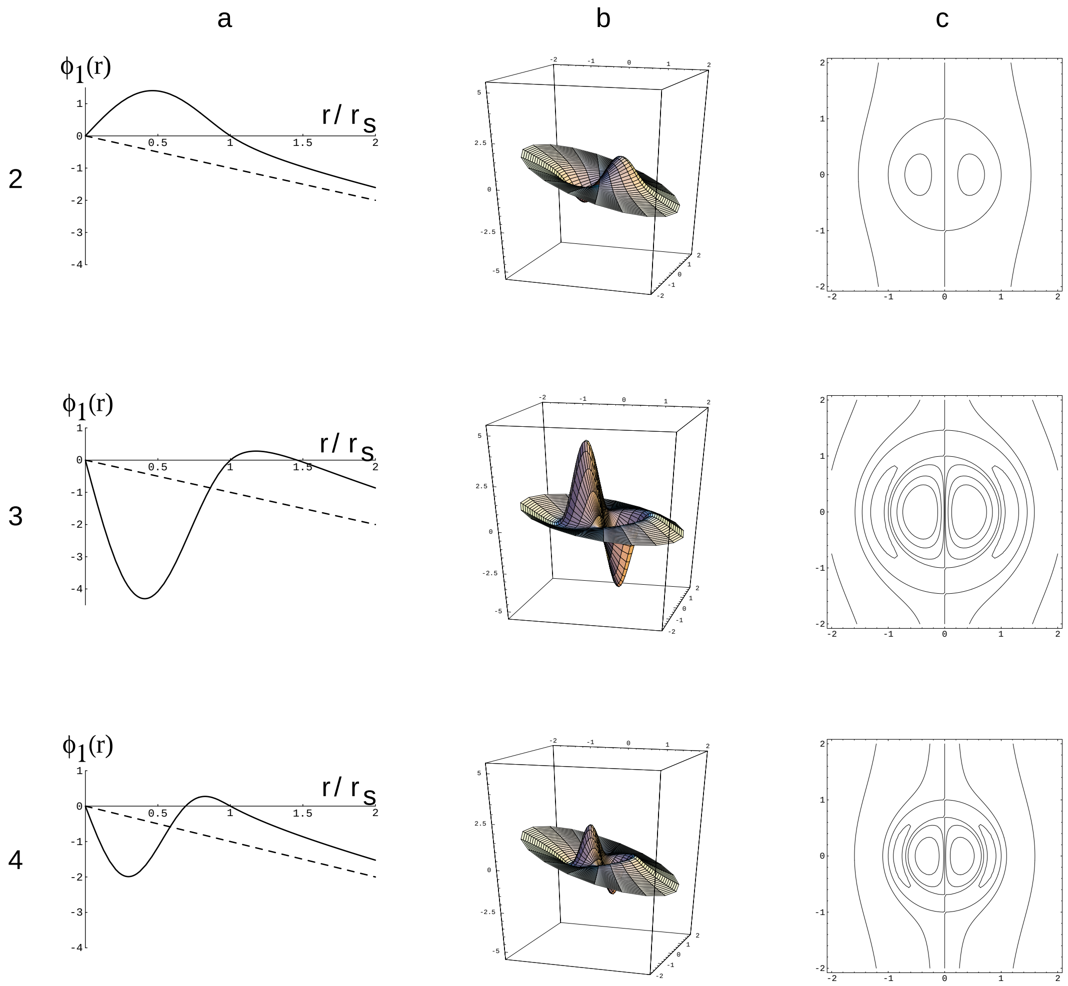


Fig.5 Zone boundaries (drawn) and zero at $r=\infty$ in $\phi_1(r)$ (dashed)



Figs.2,3,4; $k_e=5$; fig2. $k_1{}^2 r_s{}^2=-16.4$, $k_2{}^2 r_s{}^2=10.$

fig3. $k_1{}^2 r_s{}^2=-27.$, $k_2{}^2 r_s{}^2=-7.04$

fig4. $k_1{}^2 r_s{}^2=-50.$, $k_2{}^2 r_s{}^2=-15.1$