LARGE SCALE INSTABILITIES IN TWO DIMENSIONAL REDUCED MAGNETOHYDRODYNAMICS

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The question related to the stability of a sheared magnetic field in a motionless conducting fluid has been addressed since long time in the framework of Reduced MHD with resistive and viscous dissipation. Resistive instabilities [1, 2] are in fact known to develop nearby a magnetic field neutral line, where a boundary layer is formed. As a consequence, the ideal constrain of magnetic flux freezed into the fluid is relaxed and the magnetic field lines reconnect, releasing energy.

Here we present a different kind of resistive instability, occurring for higher values of the dissipative parameters and characterized by a changing in the topology of the magnetic field lines and in the formation of island-like structures of large size, while the magnetic energy is transferred to larger scales and partly converted into kinetic energy.

In a strong external magnetic field which is oriented along z, $B_z \gg B_\perp$, the Reduced MHD equations for the the magnetic flux function ψ associated to the planar magnetic field ($\mathbf{B}_\perp = \mathbf{e}_z \times \nabla\psi$) and for the stream function φ of the incompressible planar flow ($\mathbf{v}_\perp = \mathbf{e}_z \times \nabla\varphi$) read as:

$$\frac{\partial\psi}{\partial t} + [\varphi, \psi] = \eta(\nabla^2\psi - J_0) \qquad (1)$$

$$\frac{\partial\nabla^2\varphi}{\partial t} + \left[\varphi, \nabla^2\varphi\right] = \left[\psi, \nabla^2\psi\right] + \nu\nabla^4\varphi \qquad (2)$$

where $J_0 = \nabla^2\psi_0$ represents the equilibrium current density and, following a standard notation, the convective terms are written as Jacobian operators($[f, g] = \partial_x f \partial_y g - \partial_x g \partial_y f$). Equations have been normalized with respect to the characteristic length L of the sheared equilibrium, to the Alfvèn time $\tau_A = L/v_A$ and η and ν are, respectively, the inverse Lundquist number and the inverse Reynolds number. The density n is assumed to be uniform according to the incompressibility of the velocity field $\nabla \cdot \mathbf{v} = 0$. The equilibrium is given by $\psi = \cos x, \varphi = 0$.

The basic idea is to study the behaviour of ψ and φ on spatial and temporal scales much larger than the equilibrium scale and to exploit the separation of scales as a perturbative

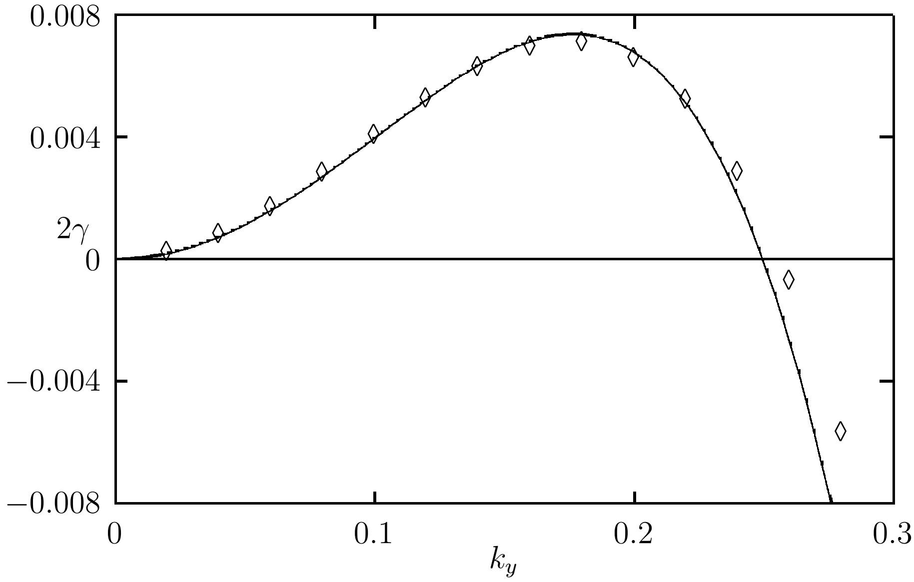

Figure 1. The linear growth rate of the large scale instability as a function of the transverse wavenumber k_y. Diamonds are obtained by a direct numerical simulation with L_x/L_y = 0.02, η = 1, ν = 0.4, N_x = 128, N_y = 1024, the full line is the expected relation dispersion

parameter. By averaging, the effect of the equilibrium is reduced to an eddy-resistivity and eddy-viscosity contribution in the large scale equations, whose sign is not positive defined ([4] for further details). As a consequence, a linear instability can set in for large enough Reynolds numbers. This instability can be thought as the MHD counterpart of the large scale hydrodynamical instability [3] which is known to develop in highly anisotropic flows.

In this paper we present analytical and numerical results concerning the development of this large scale instability near the marginal stability threshold, i.e. for values of the dissipative coefficients given by:

$$\eta = \eta_c(1-\varepsilon^2) \ , \quad \nu = \nu_c(1-\varepsilon^2) \ , \tag{3}$$

where the perturbative parameter ε^2 is fixed by the distance between η, ν and their critical values η_c, ν_c, below which the instability sets in.

We are interested in *large* scale perturbations transverse with respect to the equilibrium inhomogeneity, evolving on time scale $slower$ than that on which evolves the basic equilibrium ($\psi = \cos x, \varphi = 0$), a sheared magnetic field in a motionless conducting fluid. We then consider a set of $slow$ variables ($Y = \varepsilon y, T = \varepsilon^4 t$) besides the $fast$ ones (x, y, t) and we treat them as independent.

We solve perturbatively Eq. (1, 2) by using a standard multiple-scale technique [5, 6] and at first order in ε fields are given by:

$$\psi \ = \ cos x + \Psi^{(0)}(Y,T) + \varepsilon\Psi^{(1)}(Y,T) \tag{4}$$

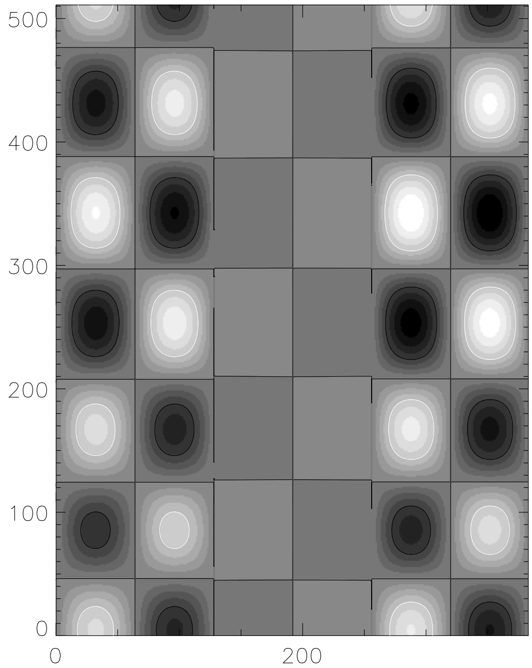

Figure 2. A comparison between the vorticity obtained by direct numerical simulation, on the left, and the vorticity evaluated at the same time by Eq. (5), on the right. Simulation parameters are: L_x/L_y = 0.02, η = 1, ν = 0.4, N_x = 128, N_y = 512

$$\varphi = -\frac{\varepsilon}{\nu_c}\frac{\partial\Psi^{(0)}}{\partial Y} sinx \; . \tag{5}$$

We notice that the stream function is linearly stable towards large scale transverse perturbations and it is simply dragged by the magnetic flux.

The evolution equation for the large scale magnetic flux $\Psi^{(0)}(Y,T)$ emerges as solvability condition at order ε^4

$$\partial_T\Psi^{(0)} = -\frac{27}{8}\eta_c\partial_{4Y}\Psi^{(0)} - 2\eta_c\partial_{YY}\Psi^{(0)} + 12\eta_c\partial_{YY}\Psi^{(0)}(\partial_Y\Psi^{(0)})^2 \tag{6}$$

while the parameter critical values $\eta_c\nu_c = 1/2$ are fixed at order ε^2.

A straightforward linear stability analysis done on Eq .(6) leads to the conclusion that the magnetic flux is unstable towards large scale transversal modes ($\Psi^{(0)} \sim exp(\gamma T + \imath KY)$), with dispersion relation given by:

$$\gamma = -\frac{27}{8}\eta_c K^4 + 2\eta_c K^2 \tag{7}$$

We integrated MHD equations (1), (2) in a rectangular box $L_x \times L_y$ imposing periodic boundary conditions, by using a standard pseudo-spectral method with resolution $N_x \times N_y$. The aspect ratio $\epsilon = L_x/L_y$ is taken smaller than one. The initial conditions are ($\psi =$

$\cos x + R_1(x,y), \varphi = R_2(x,y))$ where $R_1(x,y), R_2(x,y)$ represent random noise of amplitude 10^{-3}.

In Fig. 1 we show the growth rate of the linear instability (diamonds) as a function of the transversal wavenumber k_y for a direct numerical simulation with $\varepsilon^2 = 0.1055$, compared with (continuous line) the expected behaviour (7), evaluated for correspondent values of the parameters.

For the same value of ε, we show also in Fig. 2 a comparison between the vorticity field obtained by direct numerical simulation and the theoretical vorticity field, calculated by using Eq. (5) and the numerical data for $\Psi^{(0)}$. The central part of the plot represents the difference between the two fields, which is quite negligible.

With a correct rescaling of time and field amplitude, equation (6) reduces to the well-known Cahn-Hilliard equation which may be written in variational form:

$$\frac{\partial \Psi^{(0)}}{\partial t} = -\frac{\delta V\left[\Psi^{(0)}\right]}{\delta \Psi^{(0)}}$$

by defining the Lyapunov functional

$$V\left[\Psi^{(0)}\right] = \eta_c \int dY \left[-(\partial_Y \Psi^{(0)})^2 + (\partial_Y \Psi^{(0)})^4 + \frac{27}{16}(\partial_{YY}\Psi^{(0)})^2 \right] \tag{8}$$

This indicates the existence of steady-state solutions of (6) in a bounded domain.

Conclusion

We have here presented analytical and numerical results concerning the development and the saturation of a linear instability in the framework of two-dimensional MHD with viscous and resistive dissipation. Although these results have been obtained near the marginal stability threshold, this approach can be improved in order to reach higher values of the Reynolds numbers, comparable with that of realistic space and laboratory plasma.

References

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