BIFURCATION IN TOROIDAL PLASMAS

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Abstract

The force balance equations and the steady state plasma transport equations have bifurcation points. Here we look at the implications of simultaneous (in space and time) bifurcation of equilibrium and transport. An application of the theory to tokamaks can explain a wide variety of experimentally observed phenomena: transport bifurcations, L and H mode, formation of transport barriers in the H-mode, dithering H-mode, stability conditions for the appearance of ideal MHD and tearing modes. A previously undescribed type of localized externally driven internal MHD instability (cold bubble) is presented.

1. Bifurcation theory: interesting points

An implicit equation for a function Ψ(R, Z), with a parametric dependence in α can be written as:

$$F(\Psi, \alpha) = 0. \quad (1)$$

Here F is an operator on Ψ(R, Z) at each point in space, parametrized by α. The equations for Ψ that we shall consider in detail in this paper are of the form:

$$F(\Psi, \alpha) \equiv L(\Psi(R, Z)) + J(\Psi(R, Z), R, Z, \alpha) = 0. \quad (2)$$

in which L is any positive definite, linear operator, acting on the function Ψ(R,Z), and J(Ψ,R,Z,α) is an inhomogeneous driving term, parametrized by α.

Suppose $\Psi_0(\alpha_0)$ is a solution of (2). Then [1] for a constant δ we define the linearized responses of F to a perturbation to Ψ and α, acting on the equilibrium solution $\Psi_0(\alpha_0)$ as:

$$F_\Psi(\tilde{\Psi}) = \frac{\partial}{\partial\delta} F(\Psi_0 + \delta\tilde{\Psi}, \alpha_0)\Big|_{\delta=0} \qquad F_\alpha(\tilde{\alpha}) = \frac{\partial}{\partial\delta} F(\Psi_0, \alpha_0 + \delta\tilde{\alpha})\Big|_{\delta=0} \quad (3)$$

For each equilibrium solution Ψ(α), these are functions of (R, Z), parametrized by α. For any given solution of the equilibrium equation, we define as first order interesting points the points of the solution where $F_\Psi = 0$, or $F_\alpha = 0$, or $\alpha_\Psi = 0$. Similarly, we define second order interesting points as points were $F_{\Psi\alpha} = 0.$, or $F_{\alpha\alpha} = 0.$, or $F_{\Psi\Psi} = 0.$, or $\alpha_{\Psi\Psi}(\Psi) = 0.$ Let us also define $D \equiv (F_{\Psi\alpha})^2 - F_{\alpha\alpha} F_{\Psi\Psi}$, which is used to determine if an interesting point is critical.

Revisiting the theorem of the implicit function we classify the points of an equilibrium (stationary) solution as regular or singular: regular points have $F_\Psi \neq 0$, singular points have $F_\Psi = 0$ and $F_\alpha = 0.$ A singular point can be a double point: $F_{\alpha\alpha} \neq 0$ and D>0, or $F_{\alpha\alpha} = 0$ and D>0. A cusp is a singular point with D=0 and $F_{\alpha\alpha} \neq 0$, or $F_{\Psi\Psi} \neq 0.$ We shall call interesting points all the points where have $F_\Psi = 0$ <u>or</u> $F_\alpha = 0.$ A given equilibrium solution can have multiple interesting points, and a given point of the solution can be interesting for more than one choice of the alpha parameter. Note that interesting points that are doubly interesting, with D>0, are double points: the solution bifurcates. These definitions are local in (R,Z).

2. The Grad-Shafranov equation and its interesting points

Static equilibrium in toroidally symmetric neutral plasmas is described by the Grad-Shafranov (GS) equation:

$$\Delta^*\Psi + \mu_0 R J_t(\Psi, R, \alpha) = 0., \text{ with } \Delta^*\Psi \equiv R^2\nabla\left(\frac{1}{R^2}\nabla\Psi\right) \text{ and } J_t = Rp' + \frac{II'}{\mu_0 R} \quad (4)$$

Here, Ψ is the poloidal magnetic flux per radian, J_t is the toroidal current density, μ_0 is the vacuum permeability, I≡ RB_t , B_t is the toroidal magnetic field, p is the plasma pressure, and both p and I are functions of Ψ. The parameter α is introduced to describe any dependence of the current density profiles on variables other than Ψ, R. Note that J_t has no explicit dependence in Z. Equation (4) is an implicit equation for Ψ(R,Z), and boundary conditions are provided by the fields from external field coils, as well as limiter surfaces it there are any. We define $J(\Psi,\alpha) \equiv \mu_0 R J_t(\Psi,\alpha)$, not writing the spatial dependence explicitly, since both J and Ψ are assumed to be functions of space. For each Ψ, solution of (4), we then have: $F_\alpha = \alpha J_\alpha$; $F_\Psi = \Delta^*\Psi + \Psi J_\Psi$; $F_{\Psi\alpha} = F_{\alpha\Psi} = \alpha\Psi J_{\alpha\Psi}$; $F_{\Psi\Psi} = \Delta^*\Psi + \Psi^2 J_{\Psi\Psi} + \Psi\, J_\Psi$; $F_{\alpha\alpha} = \alpha J_\alpha + \alpha^2 J_{\alpha\alpha}$

3. The double point of the Grad-Shafranov equation with $F_{\alpha\alpha} \neq 0$.

The L and H States: Consider a plasma state, solution of the GS equation, with a maximum or minimum in J, at some point (R, Z,Ψ_{min}) and choose the parameter α to be the value of Ψ at that point. Then at that point F_α= 0, J'=0. If at that point J''>J/(Ψ-Ψ_{edge})^2, a bifurcation occurs. In a tokamak, this bifurcation can happen at the edge of the plasma. We conjecture that this is the LH bifurcation point of the Grad-Shafranov equation.

The power threshold scaling of the LH transition: Away from the bifurcation point, two isolated types of solutions exist: solutions with a minimum of J somewhere inside the plasma (a turning point), or solutions without one. In typical cases, tokamaks have low toroidal beta and the profile of the toroidal current J is dominated by the p' term. The I I' term is typically positive, so the size of the decreasing pressure gradient required to compensate the I I' contribution to J, to allow J=0, scales with B_t^2. Since p' scales with B_t, this suggests an explanation for the experimentally observed B_t scaling of the threshold condition for the LH transition.

Turning points (evolution of the plasma equilibrium in the H-phase): Consider the stability of an equilibrium solution with a J minimum inside the plasma at Ψ_{min} with $F_\Psi \neq 0$. Let the field index be the change in the value of Ψ_{min} due to a displacement of the spatial position of the J minimum, $\beta = \nabla|B_{pol}| \cdot \delta R$. We see that as long as the Ψ_{min} point is in a region of space where β_Ψ> 0, the minimum will move inwards, and the pressure pedestal moves into the plasma. In a typical tokamak with a divertor coil elongating the plasma, the divertor coil attracts the current density more the more it is closer to the edge, and the minimum moves inwards. The penetration depth of the transport barrier is therefore a function of the boundary conditions. In fact the time evolution of p, in response to the plasma transport equations, also affects the evolution of the transport barrier. At the X-point and also at points where when β_Ψ= 0, second order analysis is required to determine the stability of the solution.

A dithering transition would be associated with a situation where the boundary conditions don't allow the barrier to penetrate.

4. Transport bifurcations of the type $F_{\alpha\alpha} \neq 0$.

Let us consider next the flux surface averaged steady-state plasma transport equations. Assuming a diffusive model, with the particle flows proportional to the density gradient, the transport equations are of the form:

$$\frac{d}{d\Psi}\left(L_{11}\ n'(\Psi)\right) - S(\Psi, n, T, n', T', \alpha) = 0 \qquad (5)$$

$$\frac{d}{d\Psi}\left(L_{22}\ T'(\Psi)\right) - Q(\Psi, n, T, n', T', \alpha) = 0 \qquad (6)$$

Here n is the plasma density, T the plasma temperature, L_{11} is the particle diffusivity, L_{22} the thermal diffusivity (they are assumed to be positive, non-singular functions of n and T). S represents the particle sources and sinks, including off-diagonal terms such as L_{12} T'. Similarly Q represents the heat sources and sinks.

H-type profiles of p= n T have a pedestal: an off-axis point where p'≠0, p''=0. Assuming a flat density profile (for simplicity), at that point L_{22}' T'= Q. Then a steady state solution exists if Q has a local, off-axis maximum: Q≠0, Q'=0 and Q''<0. If nowhere inside the plasma p''=0, then Q does not have a local maximum anywhere inside the plasma. This last case would correspond to an L-mode type profile.

Anywhere in the plasma, off-axis, if T'=0, then F_T= Q- (T Q'/T'), and L_{22}T''= Q. If at that point Q'=0, Q''<0, then the heat transport equation has a double point, indicating a transport bifurcation. This may be related to confinement regimes such as supershots, PEP (in the particle balance equation), hot spots, etc…

The ohmic H mode, and the L mode: when a simultaneous bifurcation point of the equilibrium and the transport equations occurs at the same point in space and time, any small perturbation in the plasma can trigger a transition. Note that in ohmic plasmas $Q \cong \eta J^2 \cong VJ$ (η is the plasma resistivity, V is the loop voltage), so the heat transport and equilibrium equations have interesting points at the same plasma position in Ψ. Auxiliary heating decouples the shape of Q from the shape of J. If the resulting Q profile is of parabolic type, it does not have a minimum inside the plasma and therefore can not support a pressure pedestal: L-mode. Ohmic discharges can have L-mode or H-mode, depending on how hot and clean (low resistivity) the plasma edge is, and on the spatial (and temporal) structure of V. Sawtooth triggered LH transitions could be due to a J profile that already has a minimum or zero (for instance, due to a plasma current ramp): the transient localized heat pulse would put the plasma state at the transport bifurcation point.

5. The $F_\Psi = 0$ **turning points of the Grad-Shafranov equation**

If in some equilibrium solution of (3) there is a point Ψ_0 where $F_\Psi = 0$, then $J' = J/(\Psi-\Psi_0)$. If $J(\Psi_0)\neq 0$, this a logarithmic singularity near Ψ_0, implying that as this plasma state is approached the toroidal plasma current density at that point grows.

For J''<0, we have $F_{\Psi\Psi}= \Psi^2 J''< 0$, and the solution has nowhere to go without breaking the constraints imposed on it. In a typical H-mode J profile a $F_\Psi = 0$ point can be approached in the region between the minimum in J and the last closed flux surface. It can also appear inside the sawtooth inversion radius, if J has a maximum off-axis.

Symmetry breaking: growth of MHD modes: At a rational surface in a non-rotating plasma, if $F_\Psi(R,Z) = 0$ and toroidal symmetry is broken (by toroidal field coil ripple, for instance), a specific field line would acquire higher current than the others in that flux surface. Then a 3D MHD mode can grow (a precursor to the ELM or peeling mode? a precursor to sawteeth?). The surface where the 2D and 3D solutions of the plasma state equation connect is an interesting point of the equilibrium.

Tearing modes: If we choose the parameter α to be the value of $p'(\Psi)$ at each point, at a place where p'=0, we have $F_\alpha= 0$: an interesting point. If at that point $F_\Psi= 0$, with $J\neq0$, and $J''\neq0$, it is a cusp point: the equilibrium can no longer satisfy all the constraints imposed on it. We expect this to lead to a tearing mode: an ELM, if it happens near the p pedestal.

The VH mode: the VH mode is known to appear in plasmas without sawteeth. We conjecture that it is associated with a J profile that is initially very flat on axis. As the minimum of J' moves inwards, improving transport, it can move further into the plasma before the plasma state encounters other critical points (such as $J'= \infty$). Also, there are no transient heat pulses from sawtooth to flatten the p profile, creating a critical point at p'=0, near p''=0.

Strong plasma shaping (the cold bubble): An interesting point of the equilibrium can be given by a boundary condition: when the poloidal magnetic field from the external field coils has a null inside the plasma (produced by the pushing and pulling on the plasma to shape it). If that interesting point coincides with the p'=0, $J\neq0$ interesting point of the force balance equation, a singular point results. This might lead to poloidally localized confinement degradation in strongly shaped plasmas.

6. Conclusions

A theory of bifurcation in toroidal plasmas has been presented. It may explain different plasma confinement modes, and ellucidate the conditions for appearance of MHD modes in toroidal equilibria. A generic framework for the study of bifurcation in toroidal plasmas can be constructed from the study of interesting points in any given plasma state.

Acknowledgements

I would like to acknowledge useful discussions with H. Wobig, G. Spies, H. Tasso, D. Lortz, D. Biskamp, O. Kardaun and P. H. Edmonds, as well as support from F. Wagner and IPP.

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