RELAXED STATES OF A MAGNETIZED PLASMA WITH MINIMUM DISSIPATION

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Abstract

Relaxed state of a slightly resistive and turbulent magnetized plasma is obtained by invoking the Principle of Minimum Dissipation which leads to the equation $\nabla\times\nabla\times\nabla\times\mathbf{B} = \Lambda\mathbf{B}$. A solution of the this equation is accomplished using the analytic continuation of the Chandrasekhar-Kendall eigenfunctions in the complex domain. The new features of this theory is to show (i) a single fluid can relax to an MHD equilibrium which can support pressure gradient even without a long-term coupling between mechanical flow and magnetic field (ii) field reversal (RFP) in states that are not force-free.

1. Introduction

In the well-known theory of relaxation of magnetoplasma, Taylor [1] proposed that the process of relaxation is governed by the principle of minimum total magnetic energy and invariance of global magnetic helicity $K = \int_V \mathbf{A}\cdot\mathbf{B}dV$, with the relaxed state satisfying the Euler-Lagrange equation $\nabla\times\mathbf{B} = \lambda\mathbf{B}$. Such relaxed states, as envisaged by Taylor, have only zero pressure gradient.

Extensive numerical works by Sato and his collaborators have established [2] the existence of self-organized states with finite pressure, i.e. these states are governed by the force balance relation $\mathbf{j}\times\mathbf{B} = \nabla p$. Several attempts [3, 4] have been made in the past to obtain relaxed states which could support finite pressure gradient, a large number of them making use of the coupling of the flow with magnetic field. The novel feature of our work is to show that it is possible for a $single$ fluid to relax to an MHD equilibrium with a magnetic field configuration which can support pressure gradient, even without a long-term coupling between the flow and the magnetic field.

The principle of minimum dissipation rate was used for the first time by Montgomery and Phillips [5] and later by many authors [6, 7] to predict the current and magnetic field profiles of driven dissipative systems. It is also our conjecture that relaxed states could be characterized as the states of minimum dissipation rather than states of minimum energy.

We consider a closed system of an incompressible, resistive magnetofluid, without any mean flow velocity, described by the standard MHD equations in presence of a small but finite resistivity η. In presence of small scale turbulence in the system, we may expect that the rate of energy dissipation decays at a faster rate than helicity. We therefore minimize the ohmic dissipation $R = \int \eta \mathbf{j}^2 dV$ subject to the constraints of helicity $\int \mathbf{A} \cdot \mathbf{B} dV$. The variational equation is given by

$$\delta \int \left(\eta \mathbf{j}^2 + \overline{\lambda} \mathbf{A} \cdot \mathbf{B} \right) dV = 0 \tag{1}$$

where $\overline{\lambda}$ is Lagrange's undetermined multiplier. The variation can be shown to lead to the Euler-Lagrange equation

$$\nabla \times \nabla \times \nabla \times \mathbf{B} = \Lambda \mathbf{B} \tag{2}$$

where, $\Lambda = \overline{\lambda}/\eta$ is a constant. A solution of Eq. (2) can be obtained as follows

$$\mathbf{B} = \sum_{n=0}^{2} \alpha_n \mathbf{B_n}, \tag{3}$$

where α_n are constants, with at least two of them non-zero and

$$\mathbf{B_n} = \lambda \omega^n \nabla \Phi_n \times \nabla z + \nabla \times (\nabla \Phi_n \times \nabla z) \quad n = 0, 1, 2 \tag{4}$$

with $\Phi_n = J_m(\mu_n r)\ exp[i(m\theta + kz)]$, $\mu_n^2 + k^2 = \lambda^2 \omega^{2n}$, $\omega = exp(2\pi i/3)$. The solutions given in Eq. (4) are analytic continuation in the complex domain of Chandrasekhar and Kendall's solutions of the equation $\nabla \times \mathbf{B} = \lambda \mathbf{B}$. It can be easily demonstrated that the expression for $\mathbf{B}$ given in Eq. (3) is a solution of Eq. (2) with $\Lambda = \lambda^3$. A reasonable boundary condition is to assume a perfectly conducting wall, so that we obtain the boundary conditions $\mathbf{B} \cdot \mathbf{n} = 0, \ \ \mathbf{j} \times \mathbf{n} = 0 \quad \text{at} \quad r = a$. These boundary conditions suffice to fix the arbitrary constants α_2 and α_3. The magnetic fields at the boundary $r = a$ have to obey the following relation for non-trivial values of the constants α_n

$$2B_{1r} Im(B_{2\theta} B_{2z}^*) - 2B_{1\theta} Im(\omega^2 B_{2r} B_{2z}^*) + 2B_{1z} Im(\omega^2 B_{2r} B_{2\theta}^*) = 0 \tag{5}$$

The magnetic field components for the $m = 0, k = 0$ state (cylindrically symmetric state) are given by

$$\begin{aligned} B_r &= 0 \\ B_\theta &= \lambda^2 \alpha_1 \left[J_1(\lambda r) + 2Re\left(\frac{\alpha_2}{\alpha_1} \omega^2 J_1(\lambda \omega r) \right) \right] \\ B_z &= \lambda^2 \alpha_1 \left[J_0(\lambda r) + 2Re\left(\frac{\alpha_2}{\alpha_1} \omega^2 J_0(\lambda \omega r) \right) \right] \end{aligned} \tag{6}$$

It is to be noted that for the cylindrically symmetric state the boundary condition is trivially satisfied and hence does not determine λa. To get the numerical value of λ for $m \neq 0$, we solve numerically Eq. (5) and obtain $\lambda a = 3.11$ and $ka = 1.23$ as the minimum values of λa and ka for the $m = 1$ state.

The only undetermined constant in Eq. (3) is the value of α_1 which can be determined by specifying the toroidal flux Φ_z which is obtained as

$$\Phi_z = 2\pi\alpha_1\lambda a \left[J_1(\lambda a) + 2Re\left[\frac{\alpha_2}{\alpha_1}\omega J_1(\lambda\omega a)\right]\right] \tag{7}$$

A couple of dimensionless quantities that have that are useful in describing laboratory experiments are the field reversal parameter $F = B_z(a)/ < B_z >$ and the pinch parameter $\Theta = B_\theta(a)/ < B_z >$, where $< .. >$ represents a volume average.

2. Results

The values of both F and Θ at the boundary $r = a$ are evaluated and F is shown plotted against pinch ratio Θ in Fig. 1 (a). It is observed that F reverses at a value of $\Theta = 2.4$, ($\lambda a = 2.95$) whereas for the Taylor state the reversal is achieved at $\Theta = 1.2$. However, this field reversed state supports pressure gradient in constrast to the Taylor state.

The pressure profile can be obtained from the relation $\mathbf{j} \times \mathbf{B} = \nabla p$. For the $m = 0, k = 0$ state, the only nonvanishing component of the pressure gradient exists in the radial direction. The pressure profile is shown in Fig. 1 (b) for the $m = k = 0$ state with $\lambda a = 3.0$ which is the minimum energy dissipation, field reversed state.

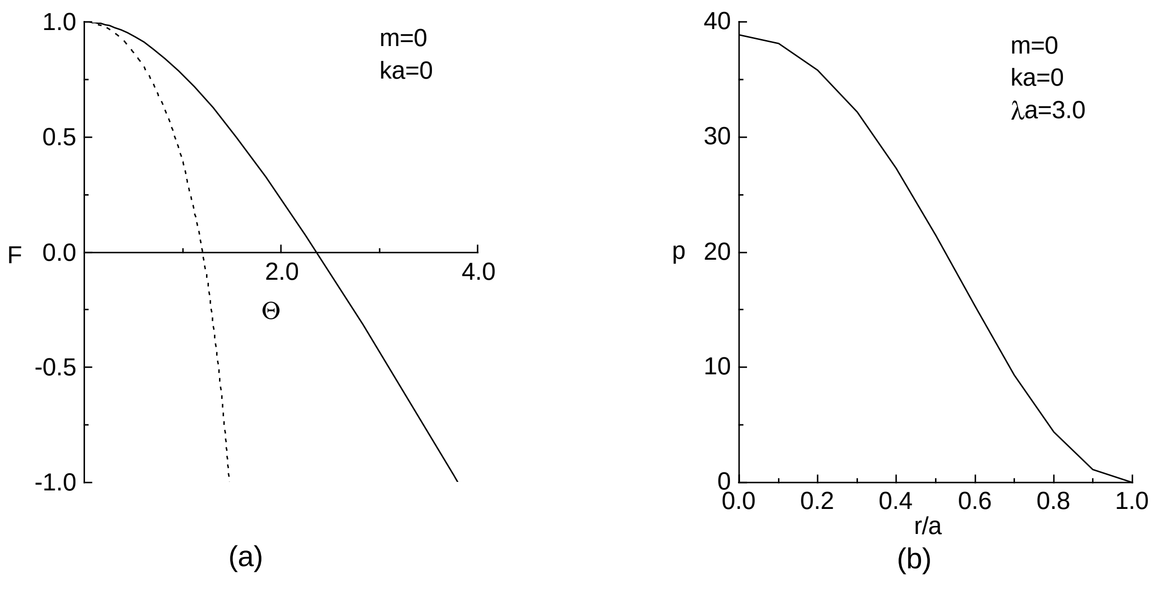

Figure 1. (a) The field reversal parameter F against the pinch parameter Θ, the field reversal occuring at $\Theta = 2.4$. The dotted curve represents the plot for the minimum energy state of Taylor. (b) The pressure profile p vs r for $\lambda a = 3.0$.

3. Conclusion

To conclude, the principle of minimum dissipation is utilized together with the constraints of constant magnetic helicity to determine the relaxed states of a magnetoplasma not driven externally. This relaxed state obtained from single fluid MHD supports pressure gradient. This establishes that a coupling between magnetic field and flow is not an essential criterion for having a non-zero pressure gradient. Further, it is shown that a non force-free state with field reversal properties can exist.

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