On the Turbulence Spectra of Electron Magnetohydrodynamics

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1. Introduction

The model of electron magnetohydrodynamics [1] (EMHD) provides a fluid description of the plasma for phenomena with frequencies above the ion gyro- and plasma frequency and below the electron gyro- ω_{ce} and plasma frequency ω_{pe}. On those time scales the ions are immobile and the plasma dynamics is governed by the electrons. In particular, the electron velocity determines the plasma current. EMHD gives the proper nonlinear framework for helicon or whistler modes and, provided the density inhomogeneity and perturbations are properly included, for electron gradient modes [2]. First studies of 2D EMHD turbulence have appeared only recently [3–5], and show a great number of interesting phenomena: the model combines both a direct cascade of energy with an inverse cascade of the square of the magnetic potential. The model contains a single length scale: the electron inertial skin depth, d_e. The turbulent spectra on scales smaller and larger than d_e are significantly different [3,5].

In the present paper, the properties of EMHD turbulence are studied from two different viewpoints. The statistical equilibrium spectra are obtained for non-dissipative turbulence on the basis of the truncated Fourier representation of the equations and of the (quadratic) conservation laws. Secondly, the spectral properties of driven and decaying dissipative EMHD turbulence are analyzed by means of the scaling symmetries of the governing equations. Numerical simulations of 2D EMHD turbulence are performed in support of the analyses.

2. 2D Electron Magnetohydrodynamics

The presence of a dominant (homogeneous) magnetic field often results in effectively 2D plasma dynamics. Taking the dominant field parallel to the z-axis, a general representation for the magnetic field is $\mathbf{B} = B_0 \{(1+b)\mathbf{e}_z + \nabla\psi \times \mathbf{e}_z\}$. The equilibrium electron density can have a small inhomogeneity, i.e. $|d\ln n_{eq}/d\ln r| \ll 1$. The density perturbations $\tilde{n}_e$ are obtained from the divergence of the electron momentum balance and Poisson's law: $\tilde{n}_e/n_0 = d_e^2(\omega_{ce}^2/\omega_{pe}^2)\nabla^2 b$, where ω_{ce} and ω_{pe} are calculated at the typical field B_0 and density n_0. The equations of 2D EMHD are now obtained from the z-component of the electron momentum balance and of its curl [2], with normalization as in Ref. [3]:

$$\frac{\partial}{\partial t}\Omega = -[b,\Omega] - [\psi,\nabla^2\psi] - [(n_e/n_0),\beta] + \mu_\nu(-\nabla^2)^\nu b, \quad \text{and} \quad \frac{\partial}{\partial t}\Psi = -[b,\Psi] + \mu_\nu(-\nabla^2)^\nu \psi, \tag{1}$$

where $\Omega = b - \Lambda d_e^2\nabla^2 b + (1 - n_{eq}(x)/n_0)$ is the generalized electron vorticity, $\Psi = \psi - d_e^2\nabla^2\psi$ is the generalized flux, and β is the normalized plasma pressure. The bracket denotes

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$[f,g] \equiv \mathbf{e}_z \cdot \nabla f \times \nabla g$. In general, the bracket of the density and pressure represents a vorticity source, and the model should be supplemented with a suitable equation of state. In the present paper the assumption $p_e = p_e(n_e)$ is made, so that this bracket vanishes. Note, that the inclusion of electron density perturbations leads to the additional factor $\Lambda = 1 + \omega_{ce}^2/\omega_{pe}^2$ in the electron vorticity. Hyperviscosity with $\nu = 3$ will be used in numerical calculations in order to allow for larger inertial ranges in the turbulence spectra.

On a closed or doubly periodic domain, the non-dissipative 2D EMHD equations possess an infinite number of invariants [3,5]. These are the energy E, the generalized helicity H, and the integrals of the generalized flux F, respectively

$$E = \frac{1}{2}\int d^2x\ [b^2 + \Lambda d_e^2 |\nabla_\perp b|^2 + |\nabla_\perp \psi|^2 + d_e^2 (\nabla_\perp^2 \psi)^2], \tag{2}$$

$$H = \int d^2x\ \Omega g(\Psi), \qquad F = \int d^2x\ f(\Psi), \tag{3}$$

where f and g are arbitrary functions of Ψ. The second term in the energy is the sum of kinetic energy $d_e^2|\nabla_\perp b|^2$ from the velocity in the plane and of the energy $(\Lambda - 1)d_e^2|\nabla_\perp b|^2$ due to the density perturbations.

3. Spectra of Non-dissipative Statistical Equilibrium

In this section, we use the Fourier representation of the fields, $b = \sum_{k_x}\sum_{k_y} \tilde{b}_{k_x,k_y} e^{i(k_x x + k_y y)}$. The series is truncated at some maximum wave vector, so that a finite dimensional system remains, which possesses the same quadratic invariants as the original infinite dimensional system, and obeys a (detailed) Liouville theorem. The methods of equilibrium statistical mechanics then are applied (see e.g. [6]), which results into the canonical probability density ρ in phase space:

$$\rho = \frac{1}{Z}\exp\left(-\alpha E - \beta H - \gamma F\right), \tag{4}$$

where the partition function Z is a normalizing constant, and α, β, γ are the Lagrange multipliers (or inverse temperatures) associated with the energy, generalized helicity, and mean square generalized potential, respectively. From the canonical distribution the following absolute equilibrium spectra are obtained as function of the wave vector $\mathbf{k} = (k_x, k_y)$

$$E(\mathbf{k}) = \frac{4\alpha k^2 + 2\gamma(1 + d_e^2 k^2)}{4\alpha[\alpha k^2 + \gamma(1 + d_e^2 k^2)] - \beta^2(1 + d_e^2 k^2)(1 + \Lambda d_e^2 k^2)}, \tag{5}$$

$$H(\mathbf{k}) = \frac{2\beta(1 + d_e^2 k^2)(1 + \Lambda d_e^2 k^2)}{4\alpha[\alpha k^2 + \gamma(1 + d_e^2 k^2)] - \beta^2(1 + d_e^2 k^2)(1 + \Lambda d_e^2 k^2)}, \tag{6}$$

$$F(\mathbf{k}) = \frac{4\alpha(1 + d_e^2 k^2)}{4\alpha[\alpha k^2 + \gamma(1 + d_e^2 k^2)] - \beta^2(1 + d_e^2 k^2)(1 + \Lambda d_e^2 k^2)}. \tag{7}$$

Note, that the convergence of the integrals of the probability density requires that α and the common denominator of the spectra are positive. For given values of the invariants E, H, and F the Lagrange multipliers are determined by summation of the spectra over the wave vector range $[k_{min}, k_{max}]$.

Some typical examples of equilibrium spectra are shown in Fig. 1. In most cases the energy spectrum is practically flat over a large range of k values, which means equipartition of energy between all modes. As there are many more modes at large k, energy will flow mostly towards small scales resulting in a normal cascade of the energy. The spectrum of the generalized potential strongly increases towards the larger scales in the range $kd_e < 1$, but is practically flat for small scales $kd_e > 1$. This suggests an inverse cascade, i.e. towards large scales, of the generalized potential which, however, is limited to scales larger than d_e. For small scales the flux is equipartitioned and the cascade direction appears to be normal. The helicity spectrum peaks both towards large and small scales with a minimum at $kd_e = 1$, suggesting an inverse cascade for scales larger than d_e and a normal cascade for smaller scales.

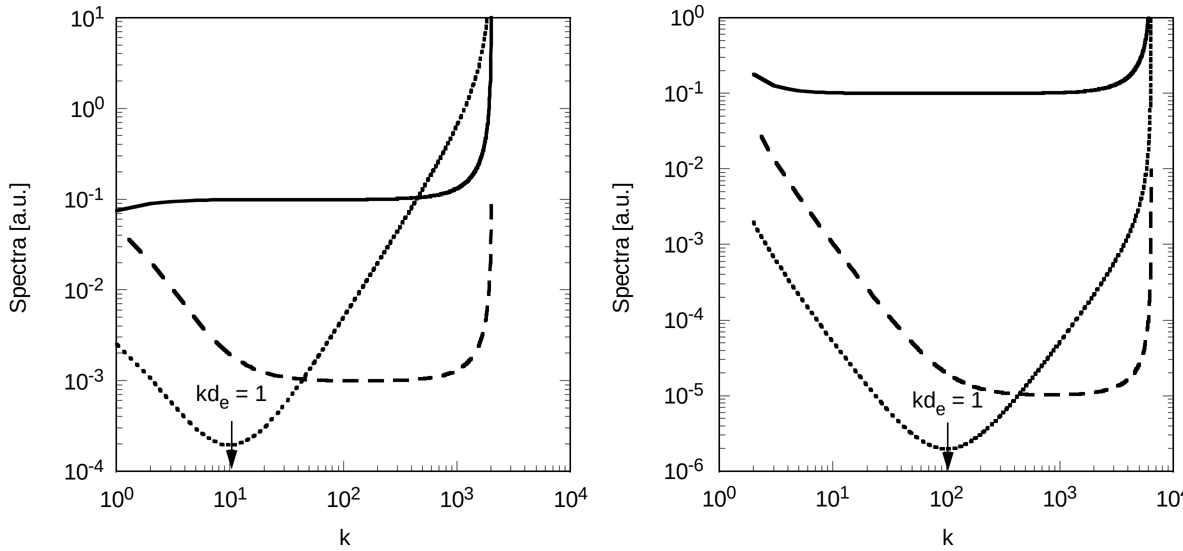

Figure 1. The ideal equilibrium spectra for E (full curves), F (dashed curves), and H (dotted curves). The left figure is for $d_e = 0.1$ with Lagrange multipliers $\alpha = \gamma = 10$, and $\beta = 1$, the right for $d_e = 0.01$ with $\alpha = \gamma = 10$, and $\beta = \sqrt{1000}$.

The canonical distribution also provides the equilibrium energy partition between $E_\psi(\mathbf{k}) \equiv \frac{1}{2}k^2(1 + d_e^2k^2)\langle|\psi_\mathbf{k}|^2\rangle$ and $E_b(\mathbf{k}) \equiv \frac{1}{2}(1 + \Lambda d_e^2k^2)\langle|b_\mathbf{k}|^2\rangle$:

$$\frac{E_b}{E_\psi} = \frac{(1 + \Lambda d_e^2k^2)[\alpha k^2 + \gamma(1 + d_e^2k^2)]}{\alpha k^2(1 + d_e^2k^2)}. \tag{8}$$

For small scales, $k^2d_e^2 \gg 1$ this becomes $E_b/E_\psi = \Lambda(1 + \gamma d_e^2/\alpha)$, while in the opposite limit $k^2d_e^2 \ll 1$ one has $E_b/E_\psi = 1 + \gamma/\alpha k^2$. In both cases the ratio is strongly dependent on the value of γ/α: for small γ/α the energies are close to equipartition, while for γ/α large and positive E_b is dominant. Dominance of E_ψ appears to require negative γ.

Numerical simulations of (dissipative) 2D EMHD turbulence show a fast evolution towards equipartition when the system is initiated with $E_b < E_\psi$. No such evolution is observed for systems initialized with $E_b > E_\psi$: rather, in this case a slow increase of E_b/E_ψ is found on the time scale of energy dissipation. This suggests that systems initialized with $E_b < E_\psi$ have $\gamma/\alpha < 1$, and vice versa.

4. Scaling Symmetries

In the limits of either $kd_e \ll 1$ or $kd_e \gg 1$, the equations of 2D EMHD are invariant under the group of transformations,

$$\mathbf{r}' = \alpha\mathbf{r}, \quad t' = \alpha^{1-\beta}t, \quad \Omega' = \alpha^{1+\beta}\Omega, \quad \Psi' = \alpha^{2+\beta}\Psi, \tag{9}$$

for arbitrary values of the parameters α and β. These scaling transformations can be used to obtain turbulent energy spectra [7].

Consider $kd_e \ll 1$: the kinetic contributions to the energy are negligible. The scaling (9) implies $b' = (r'/r)^{1+\beta} b$ and, consequently, the turbulent magnetic field perturbation over a distance r is

$$\tilde{b}(r) = r^{1+\beta} F \tag{10}$$

where F is some function of the invariants of the scaling (9). Now, we take the only invariant to be the energy dissipation rate ε, in line with the original arguments of Kolmogorov. Since the energy dissipation rate transforms as $\varepsilon' = \alpha^{3\beta+1}\varepsilon$, its invariance requires $\beta = -1/3$. Dimensional arguments then determine F to be $F = C\varepsilon^{1/3}$. The result is the following spectrum of the turbulent field perturbation and the corresponding energy spectrum $E(k)$ (integrated over angles in k-space, $E = \int E(k)dk$):

$$\langle \tilde{b}(r)\tilde{b}(r) \rangle \propto \varepsilon^{2/3} r^{4/3}, \qquad \text{or} \qquad E(k) \propto \varepsilon^{2/3} k^{-7/3}. \tag{11}$$

The case $kd_e \gg 1$ can be treated similarly. In this case the energy is dominated by the kinetic terms. The kinetic energy is $E \propto v^2$, where according to (9) the velocity scales as $v' = \alpha^\beta v$. The energy dissipation rate now transforms like $\varepsilon' = \alpha^{3\beta-1}\varepsilon$. The same arguments as above now enforce the choice of $3\beta - 1 = 0$ resulting in the spectra

$$\langle \tilde{v}(r)\tilde{v}(r) \rangle \propto \varepsilon^{2/3} r^{2/3}, \qquad \text{or} \qquad E(k) \propto \varepsilon^{2/3} k^{-5/3}, \tag{12}$$

which is identical to the Kolmogorov spectrum. These results are in agreement with the spectra as obtained in Refs. [3] and [5].

The fact that $E(k) \propto \varepsilon^{2/3}$ also has consequences for the temporal behaviour of E. After integration over the inertial range of the spectrum one has $dE/dt = -\varepsilon \propto -E^{3/2}$ which has solutions of the type

$$E/E_0 = [1 + (t/2L)\sqrt{E_0}]^{-2} \tag{13}$$

where L is a typical length scale depending on the integral of the spectral function and the size of the turbulent domain. This temporal decay is in agreement with numerical calculations with initial conditions similar to Refs. [3] and [5], provided that the calculation of the total energy is limited to the inertial range.

Acknowledgements This work was performed under the EURATOM-FOM association agreement with financial support from NWO and EURATOM.

References.

[1] A.V. Gordeev, A.S. Kingsep, L.I. Rudakov, Physics Reports **243** 215 (1994).
[2] B.N. Kuvshinov, E. Westerhof, T.J. Schep, M. Berning, Phys. Lett. A **241** (1998) 287.
[3] D. Biskamp, E. Schwarz, J.F. Drake, Phys. Rev. Letters **76** (1996) 1264.
[4] A. Celani, R. Prandi, G. Boffetta, Physica Scripta **T75** (1998) 191.
[5] D. Biskamp, E. Schwarz, A. Zeiler, A. Celani, J.F. Drake, Phys. Plasmas **6** (1999) 751.
[6] R.H. Kraichnan, D. Montgomery, Rep. Prog. Phys. **43**, (1980) 547.
[7] S.N. Gordienko, S.S. Moiseev, JETP Letters **68**, (1998) 204.