

A Magnetic Dipole Equilibrium Solution at Finite Pressure

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Abstract

A realistic equilibrium at finite pressure is derived for a plasma confined by the magnetic field of a point dipole. The low and high pressure forms of the solution are explicitly displayed. An energy principle treatment finds the equilibrium is interchange stable for arbitrary pressures and that short wavelength ballooning modes are marginally stable at high plasma pressure.

1. Introduction

Dipole confinement devices are axisymmetric toroidal systems in which the dipolar magnetic field is created by a current ring [1, 2]. All other equilibrium currents are plasma currents in the toroidal direction. All magnetic field lines are closed so that "flux" surfaces are defined as surfaces of rotation about the axis of the current ring by the closed field lines. These surfaces are also the surfaces on which the pressure is constant. The guiding centers of charged particles remain on flux surfaces and all magnetic drifts are toroidal so there is no Pfirsch-Schüller or banana transport. Of course, interest in dipoles is not limited to the laboratory since magnetic dipolar features are observed in planetary magnetospheres and star formation.

We recently presented a procedure that allows us to solve for a physically interesting and mathematically simple finite pressure plasma equilibrium confined by the magnetic field of a point dipole [3]. The low and high pressure forms of the solution were explicitly displayed; energy principle arguments find that it is stable with respect to interchange modes and that its stability improves even when the plasma pressure increases to make the confined plasma region disk shaped in the equatorial plane. Here, these results are summarized and extended by considering the ballooning stability of the equilibrium.

2. Grad-Shafranov Solutions

We consider separable solutions of the Grad-Shafranov equation

$$\nabla \cdot (R^{-2} \nabla \psi) = -4\pi dp/d\psi, \quad (1)$$

where $p = p(\psi)$ is the plasma pressure and ψ is the flux function associated with the dipole magnetic field $\mathbf{B} = \nabla\psi \times \nabla\zeta$, with the ζ toroidal angle with respect to the dipole axis and R the cylindrical radial distance from the axis of the dipole. We employ spherical coordinates r, θ , and ζ with $\mu = \cos\theta$, $R = r \sin\theta$ and seek separable solutions of the form

$$\psi = \psi_0 (R_0/r)^\alpha h(\mu), \quad (2)$$

where h is an unknown function of μ alone, that becomes $(1 - \mu^2)$ in the vacuum limit. The parameter α plays the role of an eigenvalue of the Grad-Shafranov equation and is equal to unity in the vacuum limit to recover the vacuum dipole solution $\psi \propto \sin^2\theta/r$. Plasma currents cause the parameter α to depart from unity, and since the plasma current must be in the same direction as the dipole current for equilibrium, the finite plasma pressure acts to reduce α from unity as the plasma pressure increases, as will be shown shortly.

To obtain an equilibrium dipole solution with a finite total plasma current and $h = h(\mu)$ we insert Eq. (2) into Eq. (1) and employ a pressure profile of the form

$$p = p_0(\psi/\psi_0)^{2+4/\alpha}, \quad (3)$$

with p_0 the pressure at a reference surface ψ_0 and R_0 the cylindrical radius at which the ψ_0 surface intersects the symmetry plane ($\theta = \pi/2$). The resulting nonlinear Grad-Shafranov equation for $h(\mu)$ may be written in a form particularly convenient at low plasma pressure:

$$\frac{d}{d\mu} \left[(1 - \mu^2)^2 \frac{d}{d\mu} \left(\frac{h}{1 - \mu^2} \right) \right] - (1 - \alpha)(2 + \alpha)h = -\beta_0\alpha(2 + \alpha)(1 - \mu^2)h^{1+4/\alpha}, \quad (4)$$

where $\beta_0 = 8\pi p_0/B_0^2$ is the plasma beta at the equatorial or symmetry plane. It is defined by noting that the magnitude of the magnetic field at R_0 is $B_0 = \alpha\psi_0/R_0^2$ assuming $h(\mu=0) = 1$ since the magnetic field associated with the flux function ψ is

$$\mathbf{B} = B_0(R_0/r)^{\alpha+2} [\hat{\theta}(1 - \mu^2)^{-1/2} h - \hat{r}\alpha^{-1} dh/d\mu], \quad (5)$$

where $\hat{r} = \nabla r$ and $\hat{\theta} = r\nabla\theta$ are unit vectors. Notice for the pressure profile and flux function considered here, the local plasma beta, $\beta = 8\pi p/B^2$, is independent of the spherical radial variable r so β is simply β_0 times a function of angle θ . Moreover, the boundary conditions for \mathbf{B} to be finite and up-down symmetric are $h(\mu=\pm 1) = 0$ and $dh/d\mu|_{\mu=0} = 0$.

Integration of Eq. (4) from $\mu = 0$ to $\mu = 1$ gives

$$(2 + \alpha)[(1 - \alpha)\int_0^1 d\mu h - \beta_0\alpha\int_0^1 d\mu(1 - \mu^2)h^{1+4/\alpha}] = 0. \quad (6)$$

Because $h > 0$, to satisfy Eq. (6) α should be in the range $0 < \alpha \leq 1$ (for $\alpha = -2$, $p = \text{constant}$, and $h = 1 - \mu^2$). Notice that the pressure peaks at the inner most flux surface, that is, at the location of the point dipole. Equation (6) indicates that the departure of α from unity is due to the finite β_0 of the plasma. Inserting the $\beta_0 \rightarrow 0$ result $h = 1 - \mu^2$ into Eq. (6) and assuming $\alpha \rightarrow 1$ gives the departure of α from unity: $1 - \alpha = (512/1001)\beta_0$. As a result, an analytic $\beta_0 \ll 1$ solution to (4) can be found by using $h = 1 - \mu^2$ in the terms in which h is not differentiated, and then using the boundary conditions and $h(\mu=0) = 1$ to integrate from μ to 1 [3].

Next, we consider $\beta_0 \gg 1$ to demonstrate that Grad-Shafranov solutions exist for arbitrary beta equilibrium. The low pressure solution and the lower limit on the allowed α range suggest that α decreases toward zero as β_0 increases to infinity. Consequently, we assume that $1/\beta_0 \ll \alpha(\beta_0) \ll 1$. We consider the Grad-Shafranov equation, (4), in the form

$$d^2h/d\mu^2 + \alpha(\alpha + 1)(1 - \mu^2)^{-1}h = -\beta_0\alpha(2 + \alpha)h^{1+4/\alpha}, \quad (7)$$

where we need only consider $0 \leq \mu \leq 1$ since we are interested in a solution even in μ . When $\beta_0 \gg 1$, the term $\alpha(\alpha+1)h/(1-\mu^2)$ is small everywhere. The term $\beta_0\alpha(\alpha+2)h^{1+4/\alpha}$ is large at $\mu = 0$ and rapidly decreases to zero as h decreases from $h(\mu=0) = 1$ toward $h(\mu=1) = 0$ since $\alpha \ll 1$. As a result, to lowest order we need only solve $d^2h/d\mu^2 = -2\beta_0\alpha h^{1+4/\alpha}$.

Multiplying by $dh/d\mu$ and integrating twice from $\mu = 0$, where $h(\mu = 0) = 1$, to μ gives

$$\alpha\beta_0^{1/2}\mu = \int_h^1 dx(1-x^{2+4/\alpha})^{-1/2} \xrightarrow{\mu \rightarrow 1} 1-h, \quad (8)$$

where the $\mu \rightarrow 1$ form is valid for $1 \geq \mu \gg \beta_0^{-1/2}$. To satisfy $h(\mu = 1) = 0$ requires $\alpha = 1/\beta_0^{1/2}$. For large β_0 , $h = 1 - \mu$ everywhere except in a small region $0 \leq \mu \lesssim \beta_0^{-1/2} \ll 1$ where h remains close to unity, but with a large second derivative of order $\beta_0^{1/2}$.

The preceding demonstrates that separable dipolar solutions to the Grad-Shafranov equation exist for arbitrarily large β_0 . The distance between adjacent flux surfaces at the symmetry plane $\mu = 0$ increases as β_0 increases as can be seen by realizing that as α decreases the spacing must adjust to keep $\psi \propto (R_0/r)^\alpha$ fixed. As a result, the constant ψ surfaces become more extended and localized about the symmetry plane as β_0 increases. The resulting large β_0 equilibrium resembles the accretion disk associated with star formation.

3. Stability

Energy principle arguments are employed to investigate the stability of our equilibrium. Minimizing the potential energy for a dipole field with respect to parallel displacements gives rise to a stabilizing plasma compressibility term ($\propto \gamma = 5/3$) due to the closed field lines [4]:

$$W = \int d^3r \left[\frac{Q_\perp^2}{8\pi} + \frac{B^2}{8\pi} (\nabla \cdot \xi_\perp + 2\kappa \cdot \xi_\perp) + \frac{\gamma p}{2} \langle \nabla \cdot \xi_\perp \rangle^2 - (\xi_\perp \cdot \nabla p) (\kappa \cdot \xi_\perp) \right], \quad (9)$$

where ξ_\perp is the perpendicular displacement, κ is the the curvature, $\langle \dots \rangle = v^{-1} \oint ds(\dots)/B$ with $v = \oint ds/B$, and $\mathbf{Q} = \nabla \times (\xi_\perp \times \mathbf{B})$. Writing $\xi_\perp = (\xi/R^2 B^2) \nabla \psi - \eta R^2 \nabla \zeta$ and minimizing with respect to η for interchange modes ($Q_\perp^2 = 0$) or at high toroidal mode numbers ($n \gg 1$) gives

$$W = \int d^3r \left[\frac{Q_\perp^2}{8} + \frac{2\gamma p \langle \kappa \cdot \xi_\perp \rangle^2}{1+4\pi\gamma \langle p/B^2 \rangle} - (\xi_\perp \cdot \nabla p) (\kappa \cdot \xi_\perp) \right]. \quad (10)$$

To consider interchange modes we set the line bending term to zero ($Q_\perp^2 = 0$). Variation with respect to ξ gives the general finite beta interchange stability condition:

$$2\gamma p \langle \kappa \cdot \nabla \psi / R^2 B^2 \rangle > (1 + 4\pi\gamma \langle B^{-2} \rangle) \langle dp/d\psi \rangle. \quad (11)$$

Notice that closed field lines result in plasma compressibility acting to make curvature a stabilizing influence for interchange modes. When Eq. (11) is rewritten using perpendicular pressure balance and the Grad-Shafranov equation, we recover [4]

$$(p^{-1} dp/d\psi)(v^{-1} dv/d\psi) + \gamma(v^{-1} dv/d\psi)^2 > 0, \quad (12)$$

where $v = \oint ds/B$ is the volume per unit flux at fixed ψ , which for our solution becomes

$$v = \oint ds/B = \oint d\theta r / \hat{\theta} \cdot \mathbf{B} = (R_0^3 / \alpha \psi^{1+3/\alpha}) \int_0^1 d\mu h^{1+3/\alpha}. \quad (13)$$

Using the preceding, we see that $v \propto \psi^{-1-3/\alpha}$, while from Eq. (3) $p \propto \psi^{2+4/\alpha}$. As a result, Eq. (12) gives the finite beta modified interchange stability condition for our solution to be $\gamma > 2(2 + \alpha)/(3 + \alpha)$. Because α decreases from unity towards zero as β_0 increases from zero toward infinity, we see that interchange stability is maintained at all plasma pressures.

The stabilizing influence of plasma compressibility is lost for short wavelength ballooning modes and replaced by the stabilizing influence of line bending ($Q_{\perp}^2 \neq 0$). To consider short wavelength ballooning modes we minimize Eq. (10) with respect to ξ to obtain the constraint $\langle \xi \kappa \cdot \nabla \psi / R^2 B^2 \rangle = 0$ and the reduced energy principle

$$W = \int d^3r (RB)^{-2} [(8\pi)^{-1} (\mathbf{B} \cdot \nabla \xi)^2 - \xi^2 (dp/d\psi)(\kappa \cdot \nabla \psi)] \quad (14)$$

which for our equilibrium is independent of ψ . Using $\xi = \xi_0(\mu) \cos n\zeta$, considering $\beta_0 \gg 1$, and letting $t = h^{2+3/\alpha}$ we find after considerable algebra that

$$W \propto \int_0^1 dt [4t^{3/4} (1-t)^{1/2} (1+\alpha^2-t)^{-1} (d\xi_0/dt)^2 - t^{-1/4} (1-t)^{-1/2} (1+\alpha^2-t)^{-2} (1+t)\xi_0^2].$$

Variation with respect $\xi_0 = (1-t)w(t)$ gives a hypergeometric differential equation for w in the region $1/\beta_0 = \alpha^2 \ll 1 - t \leq 1$. For $0 \leq 1-t \sim \alpha^2$ the solutions for ξ_0 are simply proportional to $(1-t)^{1/2}$ and $1 - \alpha^2(1-t)$. A marginally stable solution ($W = 0$) odd about the equatorial plane, and thereby automatically satisfying $\langle \xi \kappa \cdot \nabla \psi / R^2 B^2 \rangle = 0$, is $\xi_0 = t^{1/4} (1-t)^{1/2} F(1-a, 1-a^*; 5/4; t)$ which vanishes at $t = 0$ and matches to $(1-t)^{1/2}$ as $t \rightarrow 1$, where $a = (5+i\sqrt{7})/8$. To construct a marginally stable even solution we keep both solutions to the hypergeometric equation by writing $\xi_0 = t^{1/4} (1-t)^{1/2} F(1-a, 1-a^*; 5/4; t) - C(1-t)^{1/2} F(3/4-a, 3/4-a^*; 3/4; t)$, where the constant C is determined by the constraint $\langle \xi \kappa \cdot \nabla \psi / R^2 B^2 \rangle = 0$ and the even solution goes to a constant at $t = 0$ while matching to $(1-t)^{1/2}$ as $t \rightarrow 1$. Therefore, as the destabilizing bad curvature region become more localized about the equatorial plane at high beta, the rapid variation of the displacement in the same region makes the line bending equally stabilizing. From Eq. (14) we see that low beta are ballooning stable so we anticipate that either all beta are ballooning stable or there is a intermediate range of beta in which our equilibrium becomes ballooning unstable.

Acknowledgments

USDOE support DE-FG02-91ER-54109 at MIT and DE-FG05-80ET-53088 at UT.

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