

Equilibrium of a Gravitating Plasma in a Dipolar Magnetic Field

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Abstract

The equilibria of plasma in a dipolar magnetic field near a massive gravitating body (a star or black hole) and self gravitating plasma are considered. Analytical solutions are found that are useful for understanding the physics of plasma flows in accretion disks and star formation.

I. Introduction. Plasma equilibrium in a dipolar magnetic field is of interest for both laboratory experiments and space plasma and astrophysical applications [1, 2]. In Ref. 3 a relatively simple, arbitrary β =plasma/magnetic pressure dipolar equilibrium was found by using a separable form for the flux function in spherical coordinates with an unknown eigenvalue determined by the solution. Here we extend the results of Ref. 3 for a plasma equilibrium in a dipolar magnetic field to the case of an arbitrary β plasma in a gravitational field (see Figure).

II. Equations. Consider the equilibrium of gravitating plasma in a dipolar magnetic field

$$\mathbf{B} = \nabla\psi \times \mathbf{e}_\varphi / \rho, \quad (1)$$

where ψ is the flux function, \mathbf{e}_φ is the toroidal unit vector, and ρ the cylindrical radial distance from the axis of the dipole. From momentum balance $0 = -\nabla P - mn\nabla\Phi + (\mathbf{j} \times \mathbf{B})/c$ (where Φ is the gravitation potential; n , $P=nT$, T , \mathbf{j} , and m are the plasma density, pressure, temperature, current density, and ion mass respectively) and Ampere's equation we find

$$\nabla \cdot \left(\frac{\nabla\psi}{\rho^2} \right) = -4\pi \left[\frac{d\hat{P}(\psi)}{d\psi} + m \frac{\hat{P}(\psi)\nabla\psi}{(\nabla\psi)^2} \cdot \left(\frac{\nabla\Phi}{T(\mathbf{r})} - \nabla \int_0^s \frac{ds'}{T} \frac{\partial\Phi}{\partial s'} \right) \right] \exp \left(-m \int_0^s \frac{ds'}{T(\mathbf{r}')} \frac{\partial\Phi(\mathbf{r}')}{\partial s'} \right), \quad (2)$$

where s ($s=0$ at $\mu = \cos(\theta)=0$) is the coordinate along \mathbf{B} and $\hat{P}(\psi)$ describes the plasma pressure profile $P(\mathbf{r}) = \hat{P}(\psi) \exp(-m \int_0^s (ds'/T)(\partial\Phi/\partial s'))$. In addition, we need to consider the equation for gravitational potential

$$\nabla^2\Phi = 4\pi G_0 m \frac{\hat{P}(\psi)}{T(\mathbf{r})} \exp \left(-m \int_0^s \frac{ds'}{T(\mathbf{r}')} \frac{\partial\Phi(\mathbf{r}')}{\partial s'} \right), \quad (3)$$

where G_0 is the gravitational constant. Notice that the plasma temperature profile must be specified to close Eq. (2) and (3).

Following Ref. 3 we search for solutions of Eqs (2) and (3) in a separable form

$$\psi(\mathbf{r}) = \psi_0 H(\mu) (r_0/r)^\alpha, \quad (4)$$

where $\mu = \cos(\theta)$, $H(0) = 1$, ψ_0 and r_0 are normalization constants, with the boundary conditions $dH/d\mu|_{\mu=0} = 0$ and $H(\mu^2 \rightarrow 1) \propto 1 - \mu^2$. The magnetic field corresponding to (4) is

$$\mathbf{B}(\mathbf{r}) = B_0(r_0/r)^{2+\alpha} \left(-\mathbf{e}_\theta H(\mu)(1-\mu^2)^{-1/2} + \mathbf{e}_r (dH(\mu)/d\mu)/\alpha \right), \quad (5)$$

with $B_0 = \alpha\psi_0/r_0^2$. The parameter α plays the role of an eigenvalue of the nonlinear equilibrium equations (2), (3). It equals unity or -2 in the vacuum limit to recover the dipolar solutions $\psi_{\text{vac}} \propto (1-\mu^2)/r$ and $\psi_{\text{vac}} \propto (1-\mu^2)r^2$ describing, respectively, the flux surfaces far away from and close to the origin. To avoid the plasma gravitational condensation near the dipole axis we consider $\alpha < 0$ and the ψ function contours have the form shown in the Figure.

III. Effects of stellar or black hole gravity. We consider a plasma equilibrium near a star or black hole of mass M_s and neglect the gravity of the plasma itself. As a result we have

$$\Phi(\mathbf{r}) = -G_0 M_s / r. \quad (6)$$

Analyzing Eq. (2) with gravitational potential (6) we find that separable solution (4) is only possible for

$$T(\mathbf{r}) = T_0 t(\mu)(r_0/r) \quad \text{and} \quad \hat{P}(\psi) = P_0 (\psi/\psi_0)^{2+4/\alpha}, \quad (7)$$

where P_0 and T_0 are the normalization constants and $t(\mu = 0) = 1$. Then, from Eq. (2), (4), (6), and (7) we find

$$\frac{d^2 H}{d\mu^2} + \frac{\alpha(\alpha+1)}{(1-\mu^2)} H = -\alpha\beta H^{1+4/\alpha} \left\{ (\alpha+2) - \frac{w_g}{2t(\mu)} \right\} \exp\{w_g S(\mu)\}, \quad (8)$$

where $w_g = G_0 m M_s / (r_0 T_0)$, $\beta = 8\pi P_0 / B_0^2 = 8\pi P_0 r_0^4 / \alpha^2 \psi_0^2 = \text{plasma/magnetic pressure at the midplane } (\mu = s=0)$, and

$$S(\mu) = -\int_0^\mu d\mu' (d \ln H(\mu') / d\mu') / \alpha t(\mu'). \quad (9)$$

Notice that for the forms (4) and (7), the local beta does not depend on ψ but increases strongly with increasing s . Multiplying Eq. (8) by $1 - \mu^2$ and integrating from $\mu = 0$ to $\mu = 1$ we find

$$(2 + \alpha)[(1 - \alpha)J - \beta\alpha J_{\beta/g}] = -\alpha\beta w_g J_g / 2 \quad (10)$$

with $J = \int_0^1 d\mu H$ and

$$J_{\beta/g} = \int_0^1 d\mu (1 - \mu^2) H^{1+4/\alpha} \exp\{w_g S(\mu)\}, \quad J_g = \int_0^1 d\mu (1 - \mu^2) t^{-1} H^{1+4/\alpha} \exp\{w_g S(\mu)\}. \quad (11)$$

Assuming $t(\mu) = (H(\mu))^\tau$ ($\tau \sim 1$ is an adjustable parameter) we have $S(\mu) = (H(\mu)^{-\tau} - 1) / \alpha\tau$. For $\alpha < 0$ and $\tau > 0$ we find that $S(\mu)$ varies from $S(0) = 0$ to $S(1) = -\infty$. The case $\tau = 1/\alpha < 0$ corresponds to $T = T(\psi)$, but results in divergent integrals in (11) for small $|\alpha|$ as $\mu \rightarrow 1$ which can be avoided by the departure of T from $T \propto \psi^{1/\alpha}$ as $\mu \rightarrow 1$.

For $w_g \ll 1$ and arbitrary β , or $\beta w_g \ll 1$, gravity effects are small. We use the vacuum solution to evaluate J , J_g and $J_{\beta/g}$ and find the gravitational correction to α from (10). For $w_g \ll 1$ and arbitrary β or $\beta w_g \ll 1$, we neglect exponents in (11) and find for $\tau = 1/\alpha$

$$2 + \alpha = \pi\beta w_g / 8(1 + \beta). \quad (12)$$

For moderate gravity at low plasma pressure, $1 \ll w_g \ll 1/\beta$, $\exp\{w_g S(\mu)\}$ decreases rapidly with increasing μ . Therefore, to evaluate J_g and $J_{\beta/g}$ we expand H near $\mu=0$ to find

$$2 + \alpha = \beta^{1/2} (\pi \beta w_g / 8)^{1/2}. \quad (13)$$

For strong gravity, $w_g \gg 1$ and $\beta w_g \gg 1$, the RHS of Eq. (8) is large ($\gg 1$) at $\mu \lesssim \mu_g \sim 1/w_g \ll 1$ and small at $\mu > \mu_g$. Then, introducing $\tilde{H} = 1 - H$, we have for $\mu \ll 1$

$$d^2 \tilde{H} / d\mu^2 = 0.5 |\alpha| \beta w_g \exp\{-w_g \tilde{H} / |\alpha|\}. \quad (14)$$

From the solution of Eq. (14) we find that for $1/w_g \ll \beta \ll 1$, $|dH/d\mu| \ll 1$ and $H \approx 1$ in the region $\mu \lesssim \mu_g$. In the region $\mu \gtrsim \mu_g$, $H(\mu)$ stays close to the vacuum solution. We calculate $S(\mu)$ by using the solution of Eq. (14). Then, from Eq. (10) we find

$$2 + \alpha = \beta^{1/2} \ll 1. \quad (15)$$

For high beta, $\beta \gg 1$, we anticipate $|\alpha| \ll 1$ so that Eq. (14) is valid for $0 \leq \mu \leq 1$ and gives $\tilde{H} = |\alpha| \beta^{1/2} \mu$ for $\mu \gg \mu_g$. Therefore, using the boundary condition $H(\mu = 1) = 0$ we find

$$-\alpha = \beta^{-1/2} \ll 1. \quad (16)$$

Notice gravity affects flux surface shape but not the eigenvalue α in this limit.

IV. Effects of the plasma self gravity. Analyzing Eqs. (2) and (3) we find that separable solution (4) is only possible for

$$T(\mathbf{r}) = T_0 t(\mu) (r_0/r)^{\alpha+1} \quad \text{and} \quad \hat{P}(\psi) = P_0 (\psi/\psi_0)^{2+4/\alpha}, \quad (17)$$

which requires the following form of $\Phi(\mathbf{r})$

$$\Phi(\mathbf{r}) = -\Phi_0 \varphi(\mu) (r_0/r)^{\alpha+1}, \quad (18)$$

where Φ_0 is an unknown normalization factor and $\varphi(0) = 1$. Then from Eqs. (2), (3) we find

$$\frac{d^2 H}{d\mu^2} + \frac{\alpha(\alpha+1)}{(1-\mu^2)} H = -\alpha \beta H^{1+4/\alpha} \left\{ (\alpha+2) - \frac{(\alpha+1) w_{sg}}{2} \frac{\varphi}{t} \right\} \exp\{w_{sg} S(\mu)\}, \quad (19)$$

$$\frac{d}{d\mu} \left((1-\mu^2) \frac{d\varphi}{d\mu} \right) + \alpha(\alpha+1) \varphi = -\frac{g^2}{w_{sg}} \frac{H^{2(\alpha+2)/\alpha}}{t} \exp\{w_{sg} S(\mu)\}, \quad (20)$$

where $g^2 = 4\pi G_0 P_0 (m r_0 / T_0)^2$, and $w_{sg} = m \Phi_0 / T_0$ to be determined by the solution. Here,

$$S(\mu) = \int_0^\mu d\mu' (\varphi/t) (d \ln(\varphi H^{-(\alpha+1)/\alpha}) / d\mu'). \quad (21)$$

By integrating Eqs. (19) and (20) with the weights $1 - \mu^2$ and 1, respectively, we find

$$(2 + \alpha)[(1 - \alpha)J - \beta \alpha J_{\beta/sg}] = -\alpha(\alpha + 1)\beta w_{sg} J_{sg} / 2, \quad \alpha(\alpha + 1)J_\varphi = -g^2 J_{\varphi/sg} / w_{sg}, \quad (22)$$

where $J_\varphi = \int_0^1 d\mu \varphi$, $J_{\beta/sg} = \int_0^1 d\mu (1 - \mu^2) H^{1+4/\alpha} \exp\{w_{sg} S(\mu)\}$,

$$J_{sg} = \int_0^1 d\mu (1 - \mu^2) \frac{\varphi}{t} H^{1+4/\alpha} \exp\{w_{sg} S(\mu)\}, \quad J_{\varphi/sg} = \int_0^1 d\mu \frac{H^{2(2+\alpha)/\alpha}}{t} \exp\{w_{sg} S(\mu)\}. \quad (23)$$

Assuming $t(\mu) = H^{(\alpha+1)/\alpha} (\varphi / H^{(\alpha+1)/\alpha})^\tau$, where $\tau \sim 1$ is an adjustable parameter, we have $S(\mu) = -\{(H^{(\alpha+1)/\alpha} / \varphi)^\tau - 1\} / \tau$. For $-1 < \alpha < 0$ and $\tau > 0$ $S(\mu)$ varies from $S(0) = 0$ to $S(1) = -\infty$.

We consider the case $g \gg 1$, $\beta \gg 1$ and find $0 < -\alpha \ll 1$ and $w_{sg} \gg 1$. We use $\tilde{H} = 1 - H$ and $\tilde{\varphi} = 1 - \varphi$ to expand $S(\mu)$, and neglect small terms in Eqs. (19) and (20)

$$\frac{d^2 \tilde{H}}{d\mu^2} = -\frac{\alpha \beta w_{sg}}{2} \exp\{-w_{sg}(\tilde{H}/|\alpha| + \tilde{\varphi})\}, \quad \frac{d^2 \tilde{\varphi}}{d\mu^2} = \frac{g^2}{w_{sg}} \exp\{-w_{sg}(\tilde{H}/|\alpha| + \tilde{\varphi})\}. \quad (24)$$

Introducing $Q = w_{sg}((\alpha + 1)\tilde{H}/|\alpha| + \tilde{\varphi})$ we have

$$2 \frac{d^2 Q}{d\mu^2} = c^2 \exp\{-Q\}, \quad \text{and} \quad \int_0^{Q(\mu)} dq / \sqrt{1 - e^{-q}} = c\mu \quad (25)$$

where $c^2 = w_{sg}^2 \beta + 2g^2$ and we take into account $Q(0) = 0$, $dQ/d\mu|_{\mu=0} = 0$. We integrate Eqs. (24) by using (25). From Eq. (24) for $\tilde{H}(\mu)$ with boundary conditions $\tilde{H}(0) = 0$, $\tilde{H}(1) = 1$, and $d\tilde{H}/d\mu|_{\mu=0} = 0$, we find

$$c = |\alpha| \beta w_{sg}. \quad (26)$$

Integrating Eq. (24) for $\tilde{\varphi}(\mu)$ with $d\tilde{\varphi}/d\mu|_{\mu=0} = 0$ we find $|d\tilde{\varphi}/d\mu| \lesssim g^2/(w_{sg}c)$. We will see that for the case under consideration ($g \gg 1$ and $\beta \gg 1$) $w_{sg} \sim g$ which results in $|d\tilde{\varphi}/d\mu| \ll 1$. Then, using $\varphi \approx 1$ in the expression for J_φ , the integral relation Eq. (22) gives

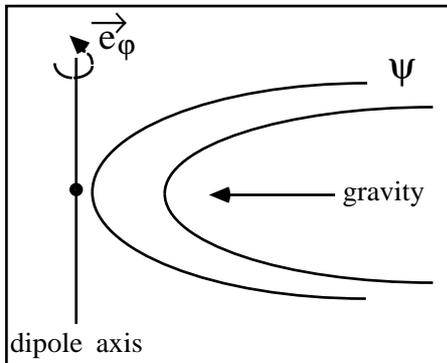
$$|\alpha| w_{sg} c = 4g^2. \quad (27)$$

From Eq. (26) and (27) we then find

$$-\alpha = \beta^{-1/2} \ll 1, \quad w_{sg} = 2g \gg 1. \quad (28)$$

V. Discussions. Interestingly, for $\beta \gg 1$ we find that $|\alpha| = \beta^{-1/2} \ll 1$ with and without the effects of gravity and toroidal plasma rotation [3, 4]. However, contour plots of the flux functions, described by $r_\psi(\mu) \propto (H(\mu))^{1/\alpha}$, are very different since a large $1/|\alpha|$ power strongly magnifies even the small differences between the $H(\mu)$ that correspond to the different cases (see [4]). Notice that the adjustable parameter τ does not enter our solutions if gravity is substantial ($w_g, w_{sg} \gg 1$) showing that the results are insensitive to the temperature profiles.

Acknowledgments. Research was supported by US DoE under grants DE-FG02-91ER-54109 at MIT and DE-FG05-80ET-53088 at UT.



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