Statistical Theory of Subcritically-Excited Strong Turbulence in Inhomogeneous Plasmas

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§1. Introduction

The study of strong turbulence in high temperature plasmas is an important issue of physics: the problems of statistical physics for systems far from thermodynamic equilibrium remain quite open, in contrast to those near thermodynamic equilibrium in which the principles that govern fluctuations (i.e., equipartition of energy, Einstein relation, fluctuation-dissipation (FD) theorem, etc.) are established.[1]

Recently, a statistical description and the extended analyses have been developed for a self-sustained strong turbulence which is caused by the subcritically excited interchange mode.[2] A Langevin equation for a dressed test mode, in which the nonlinear interactions are divided into the drag term (coherent interactions) and the random noise term (incoherent ones), is formulated. Imposing ansatz (1) of a large numbers of degrees of freedom in the turbulence (extensiveness) and (2) of the randomness of self-noise, the turbulent level and decorrelation rate of turbulence and the auto- and cross-correlation functions have been solved. The extended FD-theorem (Einstein relation) has been explicitly described for a plasma turbulence by the nonequilibrium-parameter (the gradient) of the system.[2]

In this article, we extend the analysis, including the effects of thermal fluctuations. In the treatment of thermal fluctuations, their coherent interactions with the plasma collective mode (e.g., CDIM) are represented by the collisional drags, and their incoherent interactions are considered to be a random noise of plasma temperature T. Our statistical approach for turbulence resembles to this treatment. Taking a certain test mode from turbulence modes, the coherent interactions to the mode are represented by the renormalized turbulent drags, and the incoherent interactions are considered to be a random self-noise. Combining these procedures of methods, a Langevin equation is formulated in the presence of both the thermal excitations and the inhomogeneous turbulent excitations. The solution of "stationary turbulent state" is sought for, provided that both the fluctuations are statistically independent.

§2. Basic Equation and Statistical Approach

2.1 Plasma model and basic equation

The dynamics of micro fluctuations are studied in the presence of the global inhomogeneity of the plasma pressure. Quantities that are averaged over the (y, z)-plane are denoted by the suffix 0, as p_0 and ϕ_0. $\phi = \phi_0 + \tilde{\phi}$, $J = J_0 + \tilde{J}$ and $p = p_0 + \tilde{p}$. The pressure and electrostatic potential could be inhomogeneous (i.e., inhomogeneous in the $\hat{x}$-direction) in the global scale. Parameters ∇p_0 and $\nabla_\perp^2 \phi_0$ together with Ω' represent the inhomogeneity of the system. The scale separation is introduced as $\left| p_0^{-1} \, \partial p_0/\partial t \right| << \left| \tilde{p}^{-1} \, \partial \tilde{p}/\partial t \right|$ and $\left| p_0^{-1} \nabla p_0 \right| << \left| \tilde{p}^{-1} \nabla \tilde{p} \right|$.

We consider the thermal fluctuations in the range of ω_p and the time scales between microscopic mode (CDIM) are well separated. In the thermal fluctuations, coherent parts to the microscopic CDIM are given by the collisional transport coefficients μ_c, μ_{ec} and χ_c (the ion viscosity, electron viscosity and thermal diffusivity, respectively). Incoherent parts are considered to be a random noise and expressed as $\tilde{S}_{th}$.[1] The relation between them is described by fluctuation dissipation theorem.[1]

2.2 Langevin equation for turbulent fluctuations
     A Langevin equation is deduced by use of the renormalized eddy viscosity and a random coupling model (RCM).[3] Basic equation is given by

$$\frac{\partial}{\partial t}\boldsymbol{f} + \mathcal{L}^{(0)}\boldsymbol{f} = \mathcal{N}(\boldsymbol{f}) + \tilde{\boldsymbol{S}}_{th},$$ (1)

$$\mathcal{L}^{(0)} = \begin{pmatrix} -\mu_c\nabla_\perp^2 & -\nabla_\perp^{-2}\nabla_{||} & -\nabla_\perp^{-2}\Omega'\frac{\partial}{\partial y} \\ \xi\nabla_{||} & -\mu_{ec}\nabla_\perp^2 & 0 \\ -\frac{dp_0}{dx}\frac{\partial}{\partial y} & 0 & -\chi_c\nabla_\perp^2 \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} \phi \\ J \\ p \end{pmatrix}.$$

     A projection operator $\mathcal{P}$ is introduced to divide the nonlinear interactions into the drag and others.[2] Equation (1) is written as

$$\frac{\partial}{\partial t}\boldsymbol{f} + \mathcal{L}_0\boldsymbol{f} - \mathcal{P}\mathcal{N}(\boldsymbol{f}) = (\boldsymbol{1} - \mathcal{P})\mathcal{N}(\boldsymbol{f}) + \tilde{\boldsymbol{S}}_{th}$$ (2)

($\boldsymbol{1}$ is a unit operator) where the nonlinear drag is written in an apparent linear term as

$$\mathcal{P}\mathcal{N}(\boldsymbol{f}) = \begin{pmatrix} \mu_N\nabla_\perp^2 f_1 \\ \mu_{Ne}\nabla_\perp^2 f_2 \\ \chi_N\nabla_\perp^2 f_3 \end{pmatrix} = -\begin{pmatrix} \gamma_1 f_1 \\ \gamma_2 f_2 \\ \gamma_3 f_3 \end{pmatrix}$$ (3)

and the rest part is rewritten as $\tilde{\boldsymbol{S}} = (\boldsymbol{1} - \mathcal{P})\mathcal{N}(\boldsymbol{f})$. A Langevin equation is derived as[2,3]

$$\frac{\partial}{\partial t}\boldsymbol{f} + \mathcal{L}\boldsymbol{f} = \tilde{\boldsymbol{S}} + \tilde{\boldsymbol{S}}_{th} \text{ with } \mathcal{L}_{ij} = \mathcal{L}_{ij}^{(0)} + \gamma_i\delta_{ij} ,$$ (4)

$$\gamma_{i,\,k} = -\sum_\Delta M_{i,\,kpq}M_{i,\,qkp}^*\theta_{qkp}^*\left|\tilde{f}_{1,p}^2\right| .$$ (5)

The self-noise has a much shorter correlation time (due to RCM) and is approximated to be given by the Gaussian white noise term $w(t)$ as

$$\tilde{S}_{i,\,k} = w(t)\sum_\Delta M_{i,\,kpq}\sqrt{\theta_{kpq}}\zeta_{1,\,p}\zeta_{i,\,q} .$$ (6)

2.3 Solution of Langevin equation
     To solve the Langevin eq.(4), an ansatz of large number of degrees of freedom in random modes is introduced. The general solution is formally given as

$$\boldsymbol{f}(t) = \exp\left[-\mathcal{L}t\right]\boldsymbol{f}(0) + \int_0^t \exp\left[-\mathcal{L}(t-\tau)\right]\{\tilde{\boldsymbol{S}}(\tau) + \tilde{\boldsymbol{S}}_{th}(\tau)\}\,d\tau$$ (7)

The eigenvalue of CDIM is determined by: $\det(\lambda\boldsymbol{1}+\mathcal{L}) = (\lambda+\lambda_1)(\lambda+\lambda_2)(\lambda+\lambda_3) = 0$.
     The matrix $\exp\left[-\mathcal{L}(t-\tau)\right]$ in eq.(7) is explicitly expressed as[2]

$$\exp[-\mathcal{L}(t-\tau)] = \boldsymbol{A}\exp(-\lambda_1(t-\tau)) + \boldsymbol{A}^{(2)}\exp(-\lambda_2(t-\tau)) + \boldsymbol{A}^{(3)}\exp(-\lambda_3(t-\tau))$$ (8)

2.4 Statistical average

Long-time-averaged values are calculated, where the initial condition in eq.(7) is unimportant and is neglected. We write

$$f_i(t)f_j(t) = \int_0^t d\tau \int_0^t d\tau' \{\exp[-\mathcal{L}(t-\tau)]\{\tilde{S}(\tau) + \tilde{S}_{th}(\tau)\}\}_i \{\exp[-\mathcal{L}(t-\tau')]\{\tilde{S}(\tau') + \tilde{S}_{th}(\tau')\}\}_j^*$$

where the relation eq.(6) for $\tilde{S}$ and eq.(8) should be substituted and we have the extended FD theorem of the second kind as

$$\langle f_i f_j \rangle = \frac{1}{2\lambda_1} \sum_{i', j'} A_{ii'} \{\langle \tilde{S}_{i'} \tilde{S}_{j'} \rangle + \langle \tilde{S}_{th, i'} \tilde{S}_{th, j'} \rangle\} A_{jj'}^*. \tag{9}$$

2.5 Decomposition Approximation

Terms $\langle \tilde{S}_i \tilde{S}_j \rangle$ could be given in terms of correlation functions $\langle \zeta_i \zeta_j \rangle$. The average $\langle \zeta_{i, p} \zeta_{j, q} \zeta_{i, p'} \zeta_{j', q'} \rangle$ $(p + q = k, p' + q' = k)$ is decomposed based on RCM as

$$\langle \zeta_{i,p} \zeta_{j,q} \zeta_{i,p'} \zeta_{j',q'} \rangle = \langle \zeta_{i,p}^2 \rangle \langle \zeta_{j,q} \zeta_{j',q} \rangle \delta_{pp'} \delta_{qq'} + \langle \zeta_{i,p} \zeta_{j',p} \rangle \langle \zeta_{i,q} \zeta_{j,q} \rangle \delta_{pq'} \delta_{qp'}$$

This yields relations like $\langle \tilde{S}_1 \tilde{S}_1 \rangle = 2 \sum_q M_{1, kpq}^2 \theta_{kpq} \langle \zeta_{1,p}^2 \rangle \langle \zeta_{1,q}^2 \rangle.$

We here employ the Ansatz of equivalence of correlation, i. e.,

$$\langle \zeta_i \zeta_j \rangle = \langle f_i f_j \rangle. \tag{10}$$

2.6 Auto-correlation functions and Further simplification

Taking the leading terms in eq.(9) using eq.(10), auto-correlation functions are given as

$$\begin{pmatrix} I_1(k) \\ I_2(k) \\ I_3(k) \end{pmatrix} = \frac{1}{\lambda_1} \sum_p I_1(p) \mathcal{R} \begin{pmatrix} I_1(q) \\ I_2(q) \\ I_3(q) \end{pmatrix} + \begin{pmatrix} I_{th, 1}(k) \\ I_{th, 2}(k) \\ I_{th, 3}(k) \end{pmatrix} \tag{11}$$

where the matrix $\mathcal{R}$ is expressed in terms of the matrix **A** and other coupling coefficients as $\mathcal{R}_{ij} = (1 + \delta_{j1}) M_{j, kpq}^2 \theta_{kpq} A_{ij} A_{ji}^*$ For further simplification of eq.(11), the second moments are approximately evaluated by the first one $\gamma_{i,k}$ (eq.(5)) as

$$(1 + \delta_{1j}) \sum_q M_{j, kpq} M_{j, kpq} \theta_{kpq} I_1(p) I_j(q) = C_0 \gamma_v I_j(k).$$

and we have

$$\mathbf{I} = \frac{1}{\lambda_1} \hat{\mathcal{R}} \mathbf{I} + \mathbf{I}_{th} \qquad \text{with} \qquad \hat{\mathcal{R}}_{mn} \simeq C_0 \gamma_v A_{mn} A_{nm}^*. \tag{12}$$

2.7 Extension of fluctuation dissipation theorem

With the thermal excitation term, the first law of eq.(12) is written as

$$I_1(k) = \frac{C_0 \gamma_v}{2\lambda_1} \{|A_{11}^2| I_1(k) + A_{12} A_{21}^* I_2(k) + A_{13} A_{31}^* I_3(k)\} + \frac{\mu_{vc}}{\lambda_1} \hat{T} \tag{13}$$

§3. Application to CDIM (Current Diffusive Interchange Mode)

Let us examine the connection between the thermally-excited fluctuations and the turbulent fluctuations of CDIM. Equation (13) is solved and γ_v is given in terms of normalized temperature $\bar{T}$ and the gradient parameter $\hat{G}_0$. In an explicit form, one writes as

$$\left(\gamma_v + \gamma_{vc} - \frac{C_0\gamma_v}{2} - \hat{G}_0^{3/5}\gamma_{vc}^{4/5}(\gamma_v + \gamma_{ec})^{1/5}\right)\gamma_v(\gamma_v + \gamma_{vc}) = \gamma_{vc}^3\bar{T} \qquad (14)$$

A bird's eye view of the relation $\hat{I}\left[\hat{G}_0, \bar{T}\right]$ is shown in Fig. 1 (conceptual). The cusp type manifold structure is seen. In the low temperature limit, the function $\hat{I}\left[\hat{G}_0; fixed\ \bar{T}\right]$ has a hysteresis. The upper plane represents the turbulent fluctuation level, and the lower plane corresponds to the one excited by the thermal fluctuation. The middle one satisfies the neutral condition, but is an unstable branch. Eddy damping rate γ_v (solid line) and the decorrelation rate λ_1 (dashed line) are shown as function of the gradient at low temperature for $\bar{T} = 0.003$ and $C_0 = 0.6$ in Fig. 2.
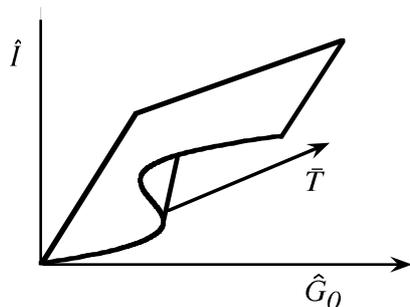

Fig. 1. *Conceptual bird eye view of the fluctua–tion level as a function of the gradient and temp–erature. Cusp – type catastrophe is constructed.*
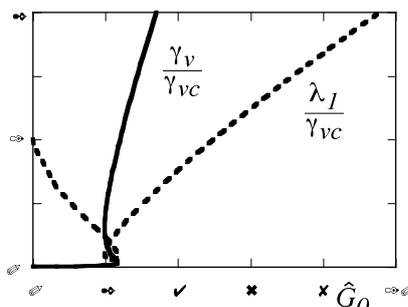
Fig. 2. *Eddy damping rate γ_v (solid line) and the decorrelation rate λ_1 (dashed line) are shown as function of the gradient at low temperature for $\bar{T} = 0.003$ and $C_0 = 0.6$.*

References

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3) See e.g., R. H. Kraichnan and D. Montgomery: Rep. Prog. Phys. **43** (1980) 547, R. H. Kraichnan: J. Fluid Mech. **41** (1970) 189, J. A. Krommes: Plasma Phys. Contr. Fusion **41** (1999) A641.