Three Dimensional Studies of a Modified Hasegawa-Wakatani Model

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The evolution of nonlinear electrostatic drift waves has received great interest in studies of confinement of hot plasmas by externally imposed magnetic fields. The interest in electrostatic drift waves stems mainly from their importance for turbulent transport of plasma across magnetic field lines.

Drift waves are linearly destabilised by either kinetic effects or by collisional resistivity of the electron motion. The latter effect is treated in the present work. The most successful model describing resistive drift waves is the Hasegawa-Wakatani model [1], which consists of two nonlinear coupled partial differential equations in the perturbations of the density n, and the electrostatic potential ϕ. The equations give a relatively simple description of a wave type which is important for naturally occurring plasmas, and their properties have been extensively studied, mainly by numerical methods, see for instance [2] and [3]. The Hasegawa-Wakatani model is derived from the ion vorticity equation, the electron continuity equation and the generalised Ohm's Law, under a number of assumptions. The model describes nonlinear electrostatic perturbations with frequencies $\omega_t \ll \omega_{ci}$, where ω_{ci} is the ion cyclotron frequency, and assumes the relative background electron density gradient to be constant $n_0(x) \propto e^{-x/L_n} \Rightarrow \frac{1}{n_0}\nabla n_0 = -\frac{1}{L_n}\hat{x}$. A Cartesian slab geometry is used with a constant magnetic field $\mathbf{B}_0 = B_0\hat{z}$. For the scale lengths of the drift wave perturbations it is assumed that $\frac{1}{L_n^2} \ll k^2 \ll \frac{1}{\lambda_D^2}$, where λ_D is the Debye length. Hence quasi-neutrality may be assumed. The $\mathbf{E} \times \mathbf{B}$-drift is the dominant drift velocity. Only the divergence of the polarisation drift is kept. Parallel ion motion is neglected. Finally, effects of finite ion temperature (such as finite Larmor radius effects) as well as electron temperature perturbations and gradients are neglected.

The omission of ion temperature dependent terms is not realistic. Furthermore, it appears that the Hasegawa-Wakatani equations do not complicate significantly by the inclusion of finite ion temperature terms under the assumption $T_i/T_e \ll 1$, which may be valid in e.g. the edge of a fusion plasma. Hence, in the present work we include finite ion temperature terms in the derivation of the equations.

The relative perturbations in number density n, electrostatic potential ϕ, and ion vorticity $\omega \ (= \nabla \times \mathbf{v}_i)$ fulfill the well-known drift ordering: $\frac{n_1}{n_0} \sim \frac{e\phi}{T_e} \sim \frac{\omega}{\omega_{ci}} \sim \frac{\rho_s}{L_n} \sim \varepsilon \ll 1$, where ρ_s is the ion gyro radius at the electron temperature given by $\rho_s = \frac{\sqrt{T_e m_i}}{eB_0} = \frac{c_s}{\omega_{ci}}$, where c_s is the sound speed. If we normalise the equations to the order unity we obtain the following set of modified Hasegawa-Wakatani equations in dimensionless variables:

$$(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla_\perp)n + \frac{\partial \phi}{\partial y} = \mathcal{C}\frac{\partial^2}{\partial z^2}(n - \phi) \tag{1}$$

$$(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla_\perp)(\nabla_\perp^2 \phi) - \theta\frac{\partial}{\partial y}\nabla_\perp^2 \phi = \mathcal{C}\frac{\partial^2}{\partial z^2}(n - \phi) + \upsilon\nabla_\perp^4 \phi \tag{2}$$

where $\theta = T_i/T_e$ and

$$\mathcal{C} \equiv \frac{T_e L_n}{\eta e^2 n_0 \omega_{ci} \rho_s L_\parallel^2}$$

The θ-containing term in (2) originates from the fluctuating ion diamagnetic drift at finite ion temperatures, and is ignored in the standard version of the Hasegawa–Wakatani equations [1]. Since, however, the standard set of Hasegawa-Wakatani equations explicitly retains the collisional part of the ion viscosity, it is consistent also to keep other terms which originate from finite ion temperatures as the θ-term in (2).

In the numerical implementation the kinematic viscosity term is insufficient to avoid high mode number noise, and is thus replaced by a so-called *hyperviscous* term [4] of the form $\nu(-1)^{p+1}\nabla_\perp^{2p}$. This term is added to both (1) and (2).

To obtain a dispersion relation for the linearly unstable drift waves we linearise the equations and assume plane wave solutions:

$$\omega^2 + i\omega \left[\mathcal{C}k_z^2 \left(1 + \frac{1}{k_\perp^2}\right) + \mu k_\perp^2 - i\theta k_y\right] - \mathcal{C}k_z^2 \left[ik_y \left(\frac{1}{k_\perp^2} - \theta\right) + \mu k_\perp^2\right] = 0 \qquad (3)$$

In the flute limit, $k_z \to 0$, we have $\omega = 0$ but also backwards propagating weakly damped waves, $\omega = -\theta k_y - i\mu k_\perp^2$. These latter wave–modes depend critically on the ion temperature. The splitting of the double root at $\omega = 0$ induced by $\theta \neq 0$ will be conspicuous for instance in numerical simulations for large times $t > 1/(\theta k_y)$, manifested physically by the "beating" between the two wave modes.

From the dispersion relation we may find the linear growth rate γ, which is dispersive and is plotted in Figure 1 as a function of k_y and k_z. The maximum growth rate for $\theta = 0.2$ is $\gamma(k_x = 0, k_y = 0.82, k_z = 0.46) \approx 0.147$, whereas for $\theta = 0$ it is $\gamma(k_x = 0, k_y = 1, k_z = 0.5) \approx 0.150$. Generally, the damping of the higher wave numbers is increased for increasing values of θ.
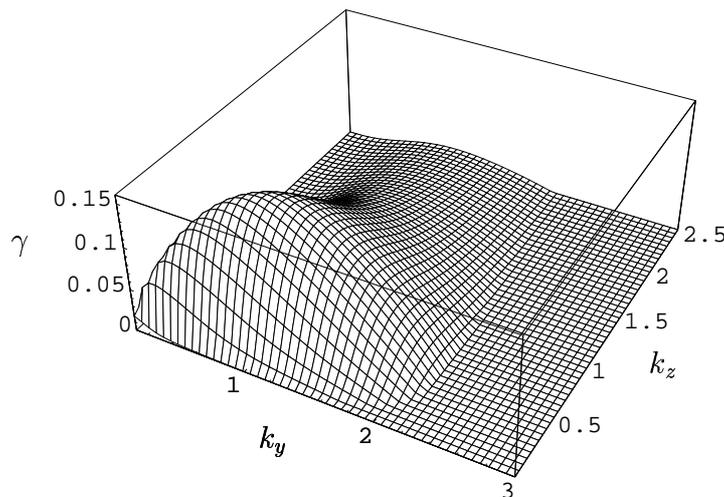

Figure 1: The linear growth rate γ, as a function of k_y and k_z , for $k_x = 0$, $\mathcal{C} = 1$, $\theta = 0.2$, $\nu = 10^{-4}$ and $2p = 6$.

The energy of the system is defined as $\mathcal{E} \equiv \frac{1}{2} \int \left[(\nabla_\perp \phi)^2 + n^2 \right] dx dy dz$ and by studying the temporal derivative of this, the sources and sinks of the system may be found. As in the case of the Hasegawa-Wakatani system the source is the background density gradient, whereas the losses are due to Ohmic resistivity and hyperviscosity. It turns out that the θ-containing term do not contribute, and do thus only redistribute the energy between modes. The Hasegawa-Wakatani equations thus form a closed set including driving in terms of the linear instability and dissipation in terms of ion viscosity.

The numerical implementation will not be discussed in detail here (details may be found in [5]), but the equations are solved in a triply periodic geometry using Fourier spectral methods and a semi-implicit scheme using the third order Stiffly Stable method [6] for the temporal integration. The parameters used in the simulation presented below were a domain size of $L_x = L_y = 12\pi$ and $L_z = 28\pi$, a spatial resolution of 96^3 Fourier modes, $dt = 2 \cdot 10^{-3}$, hyperviscosity of the order $2p = 6$ and a parameter $\nu = 10^{-4}$, $\mathcal{C} = 1$, and $\theta = 0.2$.

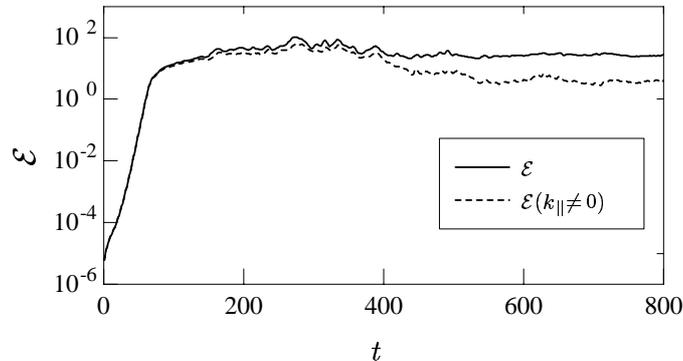The temporal evolution of the energy of the system is presented in Figure 2.

Figure 2: The evolution of the total energy $\mathcal{E}$, and the drift wave energy $\mathcal{E}(k_\parallel \neq 0)$, for the ion temperature modified Hasegawa-Wakatani model.

In the figure a distinction is made between the total energy of the system $\mathcal{E}$ and the drift wave component $\mathcal{E}(k_\parallel \neq 0)$. The energy grows exponentially due to the linear instability of the drift waves, which are initialised by low level random noise. The state of exponential growth is succeeded by a turbulent state in which nonlinear coupling dominate. In this state the major direction of energy cascading is towards smaller **k**-values and finally a state dominated by flute-like convective cells with $k_\parallel = 0$ is reached.

In Figure 3 the spectral distribution of the energy is shown in $(k_\perp, k_\parallel)$–space at a) $t = 40$ (during the exponential growth) and at b) $t = 800$ (final state with convective cell). It is clear from Figure 3.b that the energy is condensed into very low k-values. In real space this condensation is found to be in a pair of convective cells of opposite polarity.

In order to point to the effect of the θ-term we present in Figure 4 the evolution of the system for the same parameters except for $\theta = 0$, i.e. of the Hasegawa-Wakatani system. It is clear that the transition into the convective cell dominated state is delayed. The transition is actually happening at the time $t = 1000$.

The reason for the earlier state transition in the T_i modified case is that the θ-term causes a relatively larger part of the energy to be kinetic energy. Biskamp and Zeiler showed in [2] that the dominant cascading direction of the kinetic energy is from high to low
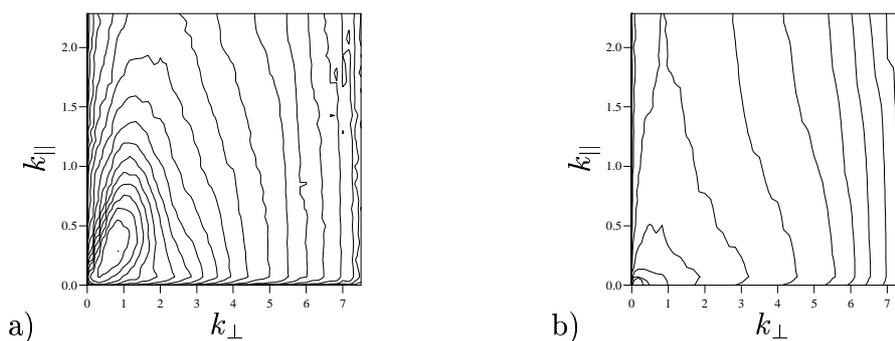
Figure 3: The energy power spectrum in $(k_\perp, k_\parallel)$–space. a) $t = 40$ (exponential growth state), b) $t = 800$ (convective cell dominated final state). The contour levels are logarithmically spaced.
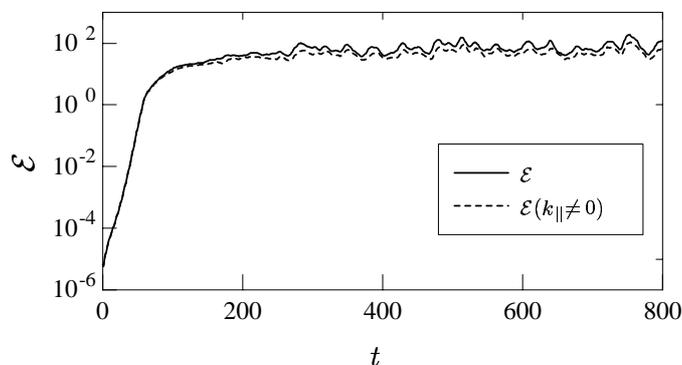

Figure 4: The evolution of the total energy $\mathcal{E}$, and the drift wave energy $\mathcal{E}(k_\parallel \neq 0)$, for the standard Hasegawa-Wakatani model.

$k_\parallel$-values. In the present simulation, however, the L_z-value was relatively high, i.e. the $k_\parallel$-values and by these also the coupling were low. In cases with smaller L_z-values the effect of the θ-term is negligible. The reason for this is that from the start of the turbulent state and onwards the nonlinear term dominates the dynamics, and we find that the norm of the nonlinear term is at least an order of magnitude larger than that of the θ-term.

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