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Propagating Ships Waves in a Rotating Tokamak Plasma

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The standard model of turbulence in tokamak plasmas is often connected with the development of various plasma instabilities. These instabilities are driven by the release of the local free energy associated with the plasma pressure, temperature and other gradients. However, if the equilibrium state of a plasma is not static, for example due to plasma rotation, then there exists another mechanism of excitation of wave motion in the plasma which is different from instabilities. The source of the wave motion in this case is some static obstacle. Such waves are known in hydrodynamics as waves produced by flow past solid bodies, e.g. ship waves. If the velocity of the flow is equal to $\mathbf{v}_0$, then the condition of ship wave excitation corresponds to the condition of the Cherenkov radiation, $\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{v}_0 = 0$, where $\omega_{\mathbf{k}}$ is the eigenfrequency and $\mathbf{k}$ is the wave vector. Thus, in the laboratory frame (or in the coordinate system related to the obstacle) the ship wave has zero frequency. We propose that ship waves excited at the edge of the rotating tokamak plasma are a source of plasma turbulence which is linked in the radial direction due to toroidicity. If the radial extention of a ship wave structure is much larger than the ion Larmor radius, then such waves can be responsible for nonlocal properties of plasma transport at L-H transitions.

We consider a toroidal rotating plasma and suppose that the plasma rotation is driven by stationary inhomogeneous electrostatic potential ϕ_0. For the case of low-β plasma of a large-aspect-ratio tokamak with concentric and circular magnetic surfaces the low-frequency, $\omega \ll \omega_{ci}$, drift modes are collisionless electrostatic oscillations and the magnetic field perturbations are negligible. The smallness of electron inertia, $\omega \ll k_{\|}v_{Te}$, allows us to neglect the charge separation and to use the quasineutrality condition instead of the Poisson equation. In this limit the electrons are thermalized along the magnetic field line and obey the Boltzmann distribution.

The fluid equations for the fluctuating potential, ϕ and parallel ion velocity, $v_{\|}$ are reduced in the above assumptions to the following set

$$\left(\omega + \frac{\mathrm{i}v_0}{r}\frac{\partial}{\partial\theta}\right)v_{\|} + \frac{\mathrm{i}v_0}{R}\left(\cos\theta\frac{\partial}{\partial\theta} - \sin\theta\right)v_{\|} = -\frac{\mathrm{i}e}{m_i qr}\left(\frac{\partial}{\partial\theta} - \mathrm{i}nq\right)\phi \tag{1}$$

$$\begin{aligned}\left(\omega + \frac{\mathrm{i}v_0}{r}\frac{\partial}{\partial\theta}\right)\left[\phi - \frac{c_s^2}{\omega_{ci}^2}\left(\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\mathrm{d}\ln n_0}{\mathrm{d}r}\frac{\partial\phi}{\partial r}\right)\right] - \\ \frac{\mathrm{i}c_s^2}{\omega_{ci}r}\left[\frac{v_0}{r} - \frac{1}{r}\frac{\mathrm{d}v_0}{\mathrm{d}r} - \frac{\mathrm{d}^2v_0}{\mathrm{d}r^2} + \frac{\mathrm{d}\ln n_0}{\mathrm{d}r}\left(\omega_{ci} - \frac{v_0}{r} - \frac{\mathrm{d}v_0}{\mathrm{d}r}\right)\right]\frac{\partial\phi}{\partial r} - \\ \frac{2\mathrm{i}c_s^2}{\omega_{ci}R}\left[\sin\theta\frac{\partial\phi}{\partial r} + \left(\frac{1}{r} + \frac{1}{2}\frac{\mathrm{d}\ln n_0}{\mathrm{d}r} - \frac{\omega_{ci}v_0}{2c_s^2}\right)\cos\theta\frac{\partial\phi}{\partial r}\right] + \frac{\mathrm{i}T_e}{\mathrm{e}qR}\left(\frac{\partial}{\partial\theta} - \mathrm{i}nq\right)v_{\|} = 0\end{aligned} \tag{2}$$

where $v_0(r) = c/B_t(\mathrm{d}\phi_0/\mathrm{d}r)$ is the poloidal rotation velocity, $q(r) = rB_t/RB_p(r)$ is the safety factor and $c_s = \sqrt{T_\mathrm{e}/m_\mathrm{i}}$ is the sound velocity.

Drift modes are known to be long wavelength along the magnetic field lines, but short wavelength in the radial direction. Let us consider the solutions of Eqs (1) and (2) for the mode localised on the rational surface $r = r_0$, defined by $m_0 - nq(r_0) = 0$, where m_0 is the poloidal mode number at the reference rational surface around which the mode is centred. It should be noted that $m_0 \gg 1$ for the drift modes of principal interest. The excitation of the m_0 mode results, due to toroidal mode coupling, in a chain excitation of the modes with poloidal numbers $m_0 \pm l$, which are located on the neighbouring rational surfaces. Supposing that the relationship of a magnetic surface to its neighbours is different for different surfaces due to the strong inhomogeneity of the equilibrium radial electric field and the poloidal rotation velocity, we search for the solutions of Eqs (1) and (2) in the form of Fourier decompositions

$$\phi(r,\theta) = \exp(\mathrm{i}m_0\theta)\sum_l \phi_l(r)\exp(\mathrm{i}l\theta)$$
$$v_{\|}(r,\theta) = \exp(\mathrm{i}m_0\theta)\sum_l v_l(r)\exp(\mathrm{i}l\theta) \tag{3}$$

Substituting Eqs (3) into Eqs (1) and (2) , we expand the coefficients which are functions of r into the Taylor series in the vicinity of the reference rational surface $r = r_0$. Keeping only the main terms, we neglect the small corrections which are much smaller than $k_\theta^2 c_s^2$ ($k_\theta = m_0/r_0$ is the local poloidal wavenumber and $\rho = c_s/\omega_{\mathrm{ci}}$ is the Larmor radius defined at the electron temperature) and the terms with second derivative of the equilibrium density, $n_0(r)$. As a result, we reduce Eqs (1) and (2) to the form

$$k_\theta V_0 \frac{r_0}{2R}\left(v_{l+1} + v_{l-1}\right) = -\frac{\mathrm{e}}{m_{\mathrm{i}} q_0 R}\left[l - k_\theta s(r-r_0)\right]\phi_l \tag{4}$$

$$k_\theta\rho^2\left(\frac{V_0}{r_0^2} - \frac{V_0'}{r_0} - V_0''\right)\phi_l - \frac{k_\theta\rho^2}{r_n}\left[\left(\omega_{\mathrm{ci}} - \frac{V_0}{r_0} - V_0'\right)\phi_l + \omega_{\mathrm{ci}}\frac{r_0}{2R}\left(\phi_{l+1}+\phi_{l-1}\right)\right] +$$
$$\left(\frac{k_\theta\rho c_s}{R} - k_\theta V_0\frac{r_0}{2R}\right)\left(\phi_{l+1}+\phi_{l-1}\right) + \frac{\rho c_s}{R}\frac{\partial}{\partial r}\left(\phi_{l+1}-\phi_{l-1}\right) - \frac{Te}{\mathrm{e}q_0 R}[l - k_\theta s(r-r_0)]v_l = 0 \tag{5}$$

Here $V_0 = v_0(r_0)$ is the local rotation velocity, $V_0' = (\mathrm{d}v_0/\mathrm{d}r)|_{r=r_0}$, $V_0'' = (\mathrm{d}^2v_0/\mathrm{d}r^2)|_{r=r_0}$, $q_0 = q(r_0)$, $r_n^{-1} = -\left(\mathrm{d}\ln n_0/\mathrm{d}r\right)|_{r=r_0}$ is the inverse spatial scale of the density inhomogeneity, $s = r_0 q_0'/q_0$ and the Doppler-shifted eigenfrequency, $\omega - k_\theta V_0$, is taken equal to zero, since we intend to study ship waves which have zero frequency in the laboratory frame.

The Strong Coupling Approximation

Eqs (4) and (5) are an infinite set of differential-difference equations for the infinite numbers of harmonics v_l and ϕ_l. To solve them, we employ an assumption which approximates the difference operator by a differential operator, thereby further reduce the set of equations to a readily solvable second-order ordinary differential equation. This is the so called strong coupling approximation. This approximation is valid for the case, when a significant number Δ_l of azimuthal harmonics are coupled through the toroidal effects. In the limit

$$1 \ll \Delta_l \ll m_0 \tag{6}$$

we replace the discrete sets of the functions $v_l(x)$ and $\phi_l(x)$, with $x = (r - r_0)/\rho$, by the continuous functions of two variables $v(x,l)$ and $\phi(x,l)$, respectively, using the ansatz

$$f_{l\pm1}(x) \longrightarrow f(x,l) \pm \frac{\partial f(x,l)}{\partial l} + \frac{1}{2}\frac{\partial^2 f(x,l)}{\partial l^2}, \quad f = \phi, v \tag{7}$$

Substituting Eq.(7) into Eqs (4) and (5) we obtain the equations

$$\frac{\partial^2 v}{\partial l^2} + 2v = -\frac{2\mathrm{e}}{m_{\mathrm{i}} q_0 r_0 k_\theta V_0} \left(l - k_\theta \rho s x\right) \phi \tag{8}$$

$$\frac{c_s k_\theta \rho}{r_n}\left[1 + \frac{r_n}{R}\left(\frac{r_0}{r_n} + \frac{r_0 V_0}{\rho c_s}\right)\right]\phi - \frac{2c_s}{R}\frac{\partial^2 \phi}{\partial x \partial l} + \frac{c_s k_\theta \rho}{2R}\left(\frac{r_0}{r_n} + \frac{r_0 V_0}{\rho c_s}\right)\frac{\partial^2 \phi}{\partial l^2} + \frac{T_{\mathrm{e}}}{\mathrm{e} q_0 R}\left(l - k_\theta \rho s x\right) v = 0 \tag{9}$$

where the following simplifying assumptions $\rho V_0 / r_0 c_s \ll 1$ and $r_0 / 2r_n \gg 1$ are made.

One can reduce the set of Eqs (8) and (9) to one equation by the method of successive approximations. Expanding the functions ϕ and v over the small parameter $r_n/R \ll 1$

$$\phi = \phi^{(0)} + \phi^{(1)} + \ldots \quad \text{and} \quad v = v^{(0)} + v^{(1)} + \ldots$$

we obtain from Eq.(9) that

$$\frac{c_s k_\theta \rho}{r_n}\phi^{(0)} = -\frac{T_{\mathrm{e}}}{\mathrm{e} q_0 R}\left(l - k_\theta \rho s x\right) v^{(0)} \tag{10}$$

Substitution of Eq.(10) into the zero-order Eq.(8) gives us an equation that can be reduced to the Hermite equation by the variable substitution $z = \sigma^{1/2}(l - k_\theta \rho s x)$ where $\sigma^2 = c_s/V_0 \cdot 2r_n/q_0^2 k_\theta^2 \rho r_0 R$. The solutions are $H_N(z)\exp(-z^2/2)$, where H_N is the Hermite polynomial of the N:th order. Differentiating only the most rapidly changing exponential part of the solution, we approximate the second derivative as follows $\partial^2 v^{(0)}/\partial z^2 \approx \sigma^2 \left(l - k_\theta \rho s x\right)^2 v^{(0)}$. We can then insert this approximation into Eq.(8) and obtain an algebraic relation between the functions v and ϕ. Putting this relation into Eq.(9) and using $\zeta = 2^{1/2}\sigma y$, we finally get a second-order ordinary differential equation of the Schrödinger type

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} + \left(E - U\frac{\zeta^2}{1+\zeta^2}\right)\phi = 0 \tag{11}$$

where

$$E = h\frac{V_0}{c_s}\left(\frac{R}{r_n} + \frac{r_0 V_0}{\rho c_s}\right)\left(2s + \frac{r_0}{2r_n} + \frac{r_0 V_0}{2\rho c_s}\right)^{-1}, \quad U = \frac{hR}{r_n}\frac{V_0}{c_s}\left(2s + \frac{r_0}{2r_n} + \frac{r_0 V_0}{2\rho c_s}\right)^{-1} \tag{12}$$

with $h = r_n/2q_0^2 R k_\theta^2 \rho r_0$. Eq.(11) defines the dispersion properties of the ship waves

Before proceeding to the solution of Eq.(11), we can see from Eqs (12) that $E > U$. Hence we look for solutions on the form,

$$\phi = \exp\left(\pm \mathrm{i}\int^{\zeta} k(x)\mathrm{d}x\right) \tag{13}$$

assuming that $k(\zeta)$ is a slowly varying function, i.e. $|\mathrm{d}k/\mathrm{d}\zeta| \ll k^2$.
Substituting Eq.(13) into Eq.(11), we obtain the dispersion equation

$$k^2(\zeta) = E - U\frac{\zeta^2}{1+\zeta^2} \tag{14}$$

Eq.(14) defines a continuous spectrum of ship wave eigenvalues $V_0(k^2)$. In apparent form, after the substitution of Eqs (12), we finally get the dispersion relation for the ship waves.

$$\frac{V_0}{c_s} = k^2 h \frac{r_0}{2\rho} - \frac{R}{2r_n}\frac{1}{1+\zeta^2} + \left[\left(k^2 h \frac{r_0}{2\rho} - \frac{R}{2r_n}\frac{1}{1+\zeta^2}\right)^2 + 2k^2 h\left(2s + \frac{r_0}{2r_n}\right)\right]^{1/2} \tag{15}$$

One can find the group velocity, $v_{\mathrm{g}} = \partial\omega/\partial k$, of these oscillations using the definition $\omega - k_\theta V_0 = 0$ of the ship waves and their dispersion Eq.(15)

$$v_{\mathrm{g}} = k_\theta c_s k h \left(2s + \frac{r_0}{2r_n} + \frac{r_0 V_0}{2\rho c_s}\right) \cdot \left[\left(k^2 h \frac{r_0}{2\rho} - \frac{R}{2r_n}\frac{1}{1+\zeta^2}\right)^2 + 2k^2 h\left(2s + \frac{r_0}{2r_n}\right)\right]^{-1/2} \tag{16}$$

It is easy to see from Eq.(15) that the product of the last two parentheses in Eq.(16) is of the order of 1. So, Eq.(16) can be reduced to $v_{\mathrm{g}} \approx k_\theta c_s k h r_0/\rho$. It should be noted here, that the group velocity, v_{g}, has the dimension of frequency, since it is defined through the dimensionless wavenumber k. Further, using the assumptions on page 3, we reduce Eq.(14) for k^2 on the form

$$k^2 = \frac{2q_0^2 R k_\theta^2 \rho V_0}{c_s}\left(\frac{R}{r_n}\frac{1}{1+\zeta^2} + \frac{r_0 V_0}{\rho c_s}\right) \tag{17}$$

In the limit cases of large ζ^2 and of small ζ^2 we get, inserting $h = r_n/2q_0^2 R k_\theta^2 \rho r_0$

$$v_{\mathrm{g}}(\zeta^2 \gg 1) \approx \frac{V_0}{q_0}\frac{r_n}{\rho^2}\left(\frac{r_0}{R}\right)^{1/2} \quad \text{and} \quad v_{\mathrm{g}}(\zeta^2 \ll 1) \approx \frac{(V_0 c_s)^{1/2}}{q_0 \rho}\left(\frac{r_n}{\rho}\right)^{1/2} \tag{18}$$

It follows from the estimates (18) that drift ship waves in a poloidally rotating tokamak plasma propagate with a group velocity, which may substantially exceed the poloidal plasma rotation velocity.

Conclusions

In summary, we study the ship waves of a drift type which are excited in a poloidally rotating tokamak plasma due to the toroidal coupling of oscillations localised at the neighbouring rational magnetic surfaces. The analysis is based on a simple model of a low-β tokamak plasma with concentric, circular magnetic surfaces and a large aspect ratio. We assume that the plasma rotation is driven by the radial electrostatic electric field typical for the H-mode of plasma confinement in tokamaks. The low-frequency electrostatic drift oscillations are investigated using the assumptions of adiabatic electrons and plasma quasineutrality.

It is shown that when employing the strong coupling approximation we find that ship waves are free propagating waves which can travel up to the plasma core.

Since ship waves are waves of zero energy in the laboratory frame, their local frequency is determined by the plasma flow velocity. The, to the dispersion equation connected, wavenumber of a ship wave propagating with the plasma rotation velocity, is obtained. Simple estimates, of the group velocity of the ship waves based on the solution of the dispersion equation, show that the group velocity may attain large values, for instance, much higher than the poloidal rotation velocity. Thus, propagating ship waves may provide a fast link between the edge and the core fluctuations.