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# Stabilization of the internal kink mode in a tokamak by toroidal plasma rotation

C. Wahlberg[1] and A. Bondeson[2]

[1]*Department of Space and Plasma Physics, EURATOM/NFR Fusion Association, Uppsala University, S-755 91 Uppsala, Sweden.*
[2]*Department of Electromagnetics, EURATOM/NFR Fusion Association, Chalmers University of Technology, S-412 96 Göteborg, Sweden.*

**Introduction** Toroidal rotation is known to have a favourable effect on the stability of external, resistive wall modes in a tokamak.[1] However, plasma rotation introduces new effects, such as centrifugal and Coriolis forces and redistribution of pressure and density, that affect the stability of all types of modes, e.g., ballooning modes and the internal kink mode. In this area relatively little is known at present. In this paper, we analyse how the ideal, internal kink mode[2] is affected by rotational velocities of the order of the sound velocity. We find that toroidal rotation of this order of magnitude provides a strong, stabilizing effect on the internal kink. The stabilizing effect comes from the nonuniform density distribution on the $q = 1$ surface caused by the centrifugal force. This makes the motion along the field lines associated with the internal kink behave similarly to the stable Brunt-Väisälä oscillations of a fluid with a stable entropy gradient in a gravitational field.[3] The drive of the internal kink must overcome this stabilizing effect. We find that, with sonic rotation and poloidal beta of order unity, the stabilization from the density nonuniformity is dominant by order of magnitude.

**Equilibrium** We consider a low-$\beta$ [$\beta_p = O(1)$] large aspect ratio ($\varepsilon = r_0/R_0 << 1$) tokamak with circular cross section. We let the plasma rotate toroidally with the rotational frequency $\Omega(r)$, and choose the ordering of $\Omega$ such that the Mach number ($M^2 = \rho\Omega^2 R_0^2/2p$) is of order unity. In terms of the Alfvén and sound frequencies, $\omega_A = B_T/R_0\rho^{1/2}$ and $\omega_s \sim \varepsilon\omega_A$, respectively, this means that $\Omega/\omega_A \sim \varepsilon$ and $\Omega/\omega_s \sim 1$. Assuming that the temperature is a flux function, the pressure ($p$) and density ($\rho$) distributions are given by $p(r,\theta)/p_0(r) = \rho(r,\theta)/\rho_0(r) = \exp(\kappa M^2)$, where $\kappa = (R/R_0)^2 - 1$. For such a density distribution, with an effective gravity $g = R\,\Omega^2$, the Brunt-Väisälä frequency[3] is

$$\omega^2 = -\frac{g}{\Gamma}\frac{d}{dR}\ln\left(\frac{p}{\rho^\Gamma}\right) = 2\left(1-\frac{1}{\Gamma}\right)\Omega^2 M^2. \qquad (1)$$

The Shafranov shift of the flux surfaces is given by $\Delta' = -r(\beta_p + l_i/2)$, where $l_i$ is defined as in the non-rotating case,[2] whereas $\beta_p$ is modified by the rotation[4]

$$\beta_p = -\frac{2}{\mu^2 r^4}\int_0^r r'^2 \frac{d}{dr'}\left(p_0 + M^2 p_0\right)dr' . \tag{2}$$

In Eq. (2), and in the following equations, $\mu$ denotes the inverse safety factor, $\mu \equiv 1/q$.

**Stability analysis** We analyse the dynamics of the $m = n = 1$ mode in the rotating equilibrium starting from the Frieman and Rotenberg equations for small perturbations of compressible, ideal MHD equilibria with mass flow[5]

$$\rho\left(\omega + i\Omega\frac{\partial}{\partial\varphi}\right)^2 \xi + \rho\Omega^2(\xi\cdot\nabla)\frac{\partial \mathbf{e}_\varphi}{\partial\varphi} + \frac{\partial \mathbf{e}_\varphi}{\partial\varphi}\nabla\cdot\left(\rho\Omega^2\xi\right) - \nabla\phi + (\mathbf{B}\cdot\nabla)\mathbf{Q} + (\mathbf{Q}\cdot\nabla)\mathbf{B} = 0. \tag{3}$$

Here, $\xi \sim \exp(-i\omega t)$ is the (Lagrangian) plasma perturbation, $\phi = -\xi\cdot\nabla p - \Gamma p\nabla\cdot\xi + \mathbf{B}\cdot\mathbf{Q}$ is the perturbed, total pressure, $\mathbf{Q} = \nabla\times\left(\xi\times\mathbf{B}\right)$ is the perturbed magnetic field, **B** is the equilibrium magnetic field, and $\Gamma$ is the adiabatic index. Assuming that the lowest order perturbation is an $m = n = 1$ mode, we expand Eq. (3) up to fourth order in $\varepsilon$. After extensive computer algebra calculations, we obtain[6] the equations for the couped $m = 1$ and $m = 2$ amplitudes:

$$\mathcal{L}_1(\xi_1) + \varepsilon^2\left[\mathcal{T}_1 + W_1\xi_1 + \frac{d}{dr}\left(r^3 W_2\frac{d\xi_1}{dr}\right) + r^2 W_3\frac{d\xi_1}{dr} + \frac{d}{dr}\left(r^3 W_4\frac{d\xi_2}{dr}\right) + \frac{d}{dr}\left(r^2 W_5\xi_2\right)\right] = 0, \tag{4a}$$

$$\mathcal{L}_2(\xi_2) + \frac{d}{dr}\left(r^3 T_1\frac{d\xi_1}{dr}\right) + r^2 T_2\frac{d\xi_1}{dr} + 2r^3\frac{d}{dr}\left[r^{-1}\left(\rho_0\Omega^2\right)'\xi_1\right] = 0, \tag{4b}$$

where $\xi_1$ and $\xi_2$ denote the $m = 1$ and $m = 2$ amplitudes, respectively, and

$$\mathcal{L}_m \equiv \frac{d}{dr}\left[r^3(m\mu - 1)^2\frac{d}{dr}\right] - r\left(m^2 - 1\right)\left(m\mu - 1\right)^2, \tag{5a}$$

$$\mathcal{T}_1 = \frac{d}{dr}\left(r^3 A_1\frac{d\xi_1}{dr}\right) + \frac{d}{dr}\left(r^3 A_2\xi_1\right) + r^2 A_3\xi_1 - \frac{d}{dr}\left[r^2\left(\rho_0\Omega^2\right)'\xi_2\right]. \tag{5b}$$

The coefficients in Eqs. (4) and (5) are given by $T_1 = (-6\mu^2 + 6\mu - 1)\Delta' + (-4\mu^2 + 10\mu - 11/2)r$, $T_2 = (6\mu^2 - 6\mu + 3)\Delta' + (2\mu^2 - 4\mu + 7/2)r$, $W_1 = [(-3\mu^2/2 + 3\mu/2 - 3/4)\Delta' r^3 - (\mu - 1)^2 r^4/2]' - (\mu^2/2 + 1/4)r^3$, $W_2 = (2\mu^2 - 2\mu + 1/4)\Delta'^2 + (-\mu^2 + \mu/2 + 1/4)\Delta' r + (-\mu^2/4 + 3\mu/4 - 7/16)r^2$, $W_3 = -3(\mu - 1)^2\Delta' r$, $W_4 = (-3\mu^2/2 + 3\mu/2 - 1/4)\Delta' + (-\mu^2/2 + 3\mu/2 - 7/8)r$, and $W_5 = (-3\mu^2/2 + 3\mu/2 - 3/4)\Delta' - 3r/8$.

Furthermore, the coefficient $A_1$ has the form ($\omega_D \equiv \omega + \Omega$)

$$A_1 = -\rho_0\omega_D^2 + \rho_0\Omega^2 M^2 - \frac{\left(2\omega_D^2 - \omega_D\Omega + \frac{1}{4}\Omega^2\right)(\rho_0\Omega)^2 + (\omega_D - 2\Omega)\rho_0\omega_D\Gamma p_0}{\Gamma p_0 - \rho_0\omega_D^2}$$

$$-\frac{\left(2\omega_D^2 + (2\mu-1)\omega_D\Omega + \frac{1}{4}(2\mu-1)^2\Omega^2\right)(\rho_0\Omega)^2 + (\omega_D + 2(2\mu-1)\Omega)\rho_0\omega_D\Gamma p_0}{(2\mu-1)^2\Gamma p_0 - \rho_0\omega_D^2}, \qquad (6)$$

with similar expressions for $A_2$ and $A_3$.[6]

We consider a situation where the safety factor $q(r) = 1/\mu(r)$ is a monotonically increasing function of $r$, with $q_0 = q(0) < 1$, and with $q'(r_1) = O(1)$ at the $q = 1$ radius $r = r_1$. To lowest order in ε, the solution to Eq. (4a) is then $\xi_1$ = const. in the region $0 < r < r_1$, and $\xi_1 \equiv 0$ for $r > r_1$. It can be shown[6] that a solvability condition for the system (4)-(5) is that the coefficient $A_1$ vanishes, to lowest order in ε, at the $q = 1$ surface. This gives the following two eigenfrequencies of the $m = n = 1$ mode

$$\omega_D^2 = \omega_s^2\left\{\frac{3}{2} \pm \left[\left(\frac{3}{2} + \Gamma^{-1}M^2\left(M^2+4\right)\right)^2 - 2M^4\Gamma^{-2}(\Gamma-1)\right]^{1/2} + \Gamma^{-1}M^2\left(M^2+4\right)\right\}, \qquad (7)$$

where $\omega_s^2 = \Gamma p_0/\rho_0 R_0^2$. For $\Gamma > 1$, both of the roots above are real and positive and represent stable oscillations. For small rotational frequencies, $M^2 << 1$, the potential instability [minus sign in Eq. (7)] can be approximated by $\omega_D^2 \approx 2\omega_s^2(\Gamma - 1)M^4/3\Gamma^2 = (\Omega^2 M^2/3)(1 - 1/\Gamma)$, which, apart for numerical factors, is the Brunt-Väisälä frequency (1). It is this solution for $\omega_D^2$ that connects to the $m = n = 1$ instability as $\Omega \to 0$, and we conclude that rotational frequencies of order $\Omega/\omega_A \sim \varepsilon$ transform the kink instability into a stable oscillation rotating relative to the plasma at a frequency of order $\omega_D \sim \Omega M(1 - 1/\Gamma)^{1/2}$. Our interpretation is that the internal kink is stabilized because the parallel motion, which is needed to keep $\nabla \cdot \xi$ sufficiently small, now gives a large positive contribution to the potential energy because of the stable density gradient along the field lines from the centrifugal force. For strong rotation, $M^2 >> 1$, the Doppler shifted frequency instead becomes $(\omega_D/\omega_s)^2 \approx (1 - 1/\Gamma)$.

If the rotation frequency is smaller by an additional factor $\varepsilon^{1/2}$, i.e. $M \sim \varepsilon^{1/2}$ and $\Omega/\omega_A \sim \varepsilon^{3/2}$, the oscillation frequency becomes a factor $\varepsilon$ smaller than $\omega_s$, i.e. $\omega_D/\omega_A \sim \varepsilon^2$. This is the same ordering as for the growth rate of the internal kink in a non-rotating plasma,[2] and in this regime the stabilizing effect of the rotation competes with the usual drive from the internal kink instability. In terms of the Bussac growth rate ($\gamma_B$),[2] the following condition for stability can be derived from Eqs. (4) in this regime:

$$\Omega^2 M^2 (1 - 1/\Gamma) > 3\gamma_B^2. \qquad (8)$$

(The factor 3 multiplying $\gamma_B^2$ represents the Pfirsch-Schlüter enhancement of inertia $1 + 2q^2$ in toroidal geometry.) A condition similar to Eq. (8) has been derived by Waelbroeck.[4] We point out, however, that in our analysis there is no contribution from the rotation to the right hand side of Eq. (8) in the regime $\Omega/\omega_A \sim \varepsilon^{3/2}$.

**Conclusions** Toroidal rotation provides a strongly stabilizing effect on the internal kink instability in a tokamak. We have analyzed the problem by a large aspect ratio, computer algebra expansion of the compressible, ideal MHD equations,[5] using the ordering $\beta_p \sim O(1)$ and $\omega/\omega_A \sim \Omega/\omega_A \sim \varepsilon$. We find that toroidal rotation transforms the internal kink instability into a stable Brunt-Väisälä oscillation, with a Doppler shifted frequency $\sim \Omega M\,(1 - 1/\Gamma)^{1/2}$ ($M$ is the sonic Mach number). For slower rotation, $\Omega/\omega_A \sim \varepsilon^{3/2}$, the stabilization is weaker, and we find a stability condition (8) involving the rotation at $q = 1$, and the Bussac growth rate. A similar stability condition has recently been derived by Waelbroeck.[4] The present analysis offers a more formal derivation by a strict aspect ratio expansion and also identifies the physical basis of the stabilization as the stable density distribution in response to the centrifugal force.

Finally we discuss the stability criterion (8) for tokamak experiments. The rotation needed to stabilize the internal kink is predicted to be of order $\varepsilon^{3/2}v_A$. With $\varepsilon \approx 0.1$ for the $q = 1$ surface, rotation speeds of this magnitude frequently occur in tokamaks with unbalanced neutral injection. However, the low-beta ordering is not always applicable to such experiments, where the global poloidal beta is usually considered as $O(1/\varepsilon)$. Experimentally, a more relevant ordering would be $p_0 = O(\varepsilon B_0^2)$, but still maintain the poloidal beta at $q = 1$ [defined in Eq. (2)] as $O(1)$, so that $M \sim \Omega/\omega_A\varepsilon^{1/2}$. Our analysis (which is not completely rigorous in this case) then predicts a scaling $\varepsilon^{5/4}\omega_A$ for the necessary rotation frequency. Depending on the numerical factors involved, this may well be in a relevant range for many observations of sawtooth stabilization. Evidently, it would be very desirable to compare experimental data with numerical stability calculations that incorporate the full effects of rotation and shaping.

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