Large eddy simulation for magnetohydrodynamics

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1 Introduction

The complete description of magnetohydrodynamic (MHD) turbulence requires the resolution of a range of scales which is known to increase very rapidly with the turbulent intensity. Direct numerical simulations are thus restricted to moderately turbulent plasmas. A typical measure of the MHD turbulence is given by the magnetic Reynolds number $R_m = v^\star l^\star/\eta_0$, where $v^\star$ is a typical turbulent velocity, $l^\star$ is the typical length scale of the system and η_0 is the magnetic resistivity. For example, the state-of-the-art fully resolved simulation of three dimensional MHD turbulence has been recently performed by Biskamp and Müller [1, 2] for $R_m \approx 6500$. Typical values of R_m in the Sun range between 10^8 and 10^{12}.

However, in many cases, the detailed description of every excited mode is unnecessary to understand the large scale behaviour of a turbulent plasma. In the context of fluid turbulence, this has prompted the derivation of large eddy simulations (LES). This numerical technique is based on the application of a spatial filter to the Navier-Stokes equation[3, 4, 5]. The resulting equation can then be simulated using a coarser grid since the fluctuations with small characteristic scales are filtered out. However, the LES equation contains an unknown subgrid scale stress tensor that needs to be modelled. This term accounts for the effects of the unresolved small scales on the resolved scales. It is usually approximated by a turbulent viscosity.

The possibility of using a similar technique for numerical studies of MHD turbulence is investigated. In addition to a turbulent viscosity term, one can in a plasma expect that the small scales will also affect the large scales through a turbulent resistivity term [6].

In the next sections, we discuss the possibility of adapting modern approaches developed for LES in fluid turbulence to the case of MHD turbulence. In particular, the dynamical procedure[7] used for deriving self-calibrated models for the subgrid scale tensors is introduced and is illustrated for homogeneous, decaying MHD turbulence.

2 The equations

The equation of incompressible magnetohydrodynamics, written in the usual units, read

$$\partial_t u_i = -\partial_j (u_j u_i - b_j b_i) - \partial_i p + \nu_0 \Delta u_i \tag{1}$$

$$\partial_t b_i = -\partial_j (u_j b_i - u_i b_j) + \eta_0 \Delta b_i \tag{2}$$

where ν_0 is the molecular viscosity. The magnetic Prandtl number is defined by $P_m = \eta_0/\nu_0$. Both the velocity and magnetic fields are divergence free : $\partial_i u_i = 0 = \partial_i b_i$. The large-scale velocity and magnetic fields described by an LES are usually regarded as the convolution between the original fields and a filter that smoothes the high wavenumber (i.e., short wavelength) structures [3]:

$$\overline{u}_i(x) = \int G_r(x-y)\, u_i(y)\, dy, \tag{3}$$

where G_r represents the filter kernel in space. The resulting equations reads

$$\partial_t \overline{u}_i = -\partial_j (\overline{u}_j \overline{u}_i - \overline{b}_j \overline{b}_i) - \partial_i \overline{p} + \nu_0 \triangle \overline{u}_i - \partial_j \tau^u_{ji} \tag{4}$$

$$\partial_t \overline{b}_i = \partial_j (\overline{u}_i \overline{b}_j - \overline{u}_j \overline{b}_i) - \eta_0 \triangle \overline{b}_i - \partial_j \tau^b_{ji} \tag{5}$$

where two unknown terms enter the LES equation and need to be modelled:

$$\tau^u_{ij} = (\overline{u_i u_j} - \overline{b_i b_j}) - (\overline{u}_i \overline{u}_j - \overline{b}_i \overline{b}_j) \tag{6}$$

$$\tau^b_{ij} = (\overline{u_i b_j} - \overline{u_j b_i}) - (\overline{u}_i \overline{b}_j - \overline{u}_j \overline{b}_i) \tag{7}$$

Detail motivations for the models will be reported in a longer paper. However, basic principles have to guide the modelling procedure (tensorial invariance, symmetries, ...). By using these guidelines, we have constructed the following model:

$$\tau^u_{ij} = \overline{\Delta}^2 C_1 |\overline{S}^u| S^u_{ij} + \overline{\Delta}^2 C_2 |\overline{S}^b| S^b_{ij} \tag{8}$$

$$\tau^b_{ij} = \overline{\Delta}^2 f (D_1 W^b_{ij} + D_2 W^u_{ij}) \tag{9}$$

Here $\overline{\Delta}^2$ is the mesh size, $S^u_{ij} = (\partial_i \overline{u}_j + \partial_j \overline{u}_i)/2$, $W^u_{ij} = (\partial_i \overline{u}_j - \partial_j \overline{u}_i)/2$, $S^u_{ij} = (\partial_i \overline{b}_j + \partial_j \overline{b}_i)/2$, and $W^u_{ij} = (\partial_i \overline{b}_j - \partial_j \overline{b}_i)/2$. Finally $f^2 = |2W^u_{ij} W^b_{ij}|$, $|\overline{S}^u| = |2S^u_{ij} S^u_{ij}|^{1/2}$ and $|\overline{S}^b| = |2S^b_{ij} S^b_{ij}|^{1/2}$. This model contains four unknown coefficients C_1, C_2, D_1 and D_2. These coefficients have been computed by an optimisation procedure (dynamic procedure) that has been described in details in the literature [7, 8, 9, 10].

3 Numerical results

In order to estimate the validity of the model presented in the previous section we have performed a low magnetic Reynolds number DNS with 96^3 modes of homogeneous decaying turbulence in a cubic box $l_x = l_y = l_z = 10\pi$. Without magnetic field, this simulation corresponds to the classical. The parameters of the simulations are $\nu_0 = \eta_0 = 0.1$.

The spectra for the velocity and magnetic fields are shown on the Figures 1 & 2 respectively. In each case, the same figure is shown with linear (top) and logarithmic (bottom) axes. The solid lines represents the results of the DNS. The circles correspond to a 24^3 LES with model (8-9) and the crosses correspond to a 24^3 LES without any model. Clearly, the LES without model does not perform satisfactorily and a piling up of energy is observed in the high wave vector range. This is typical of an unresolved simulation.

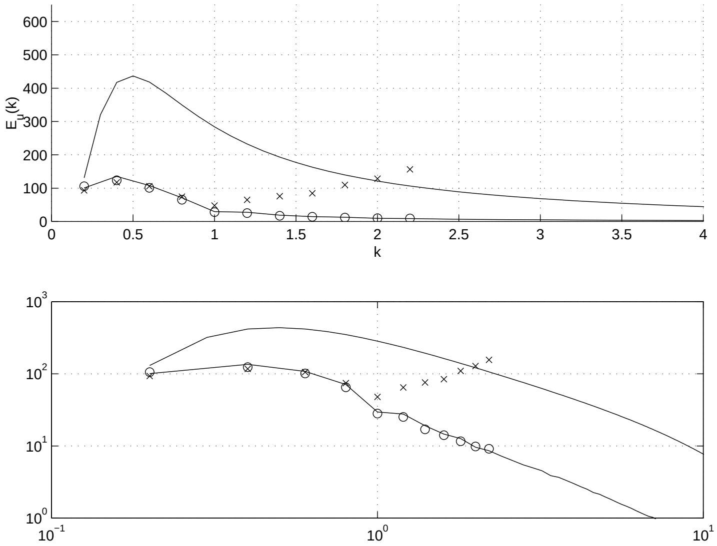

Figure 1 : velocity spectrum

On the contrary the LES with model (8-9) is able to reproduce the correct energy spectra. It must be stressed here that the LES is much cheaper that the DNS. For instance, the present LES simulation uses 100 times less CPU time than the DNS and 25 times less memory.

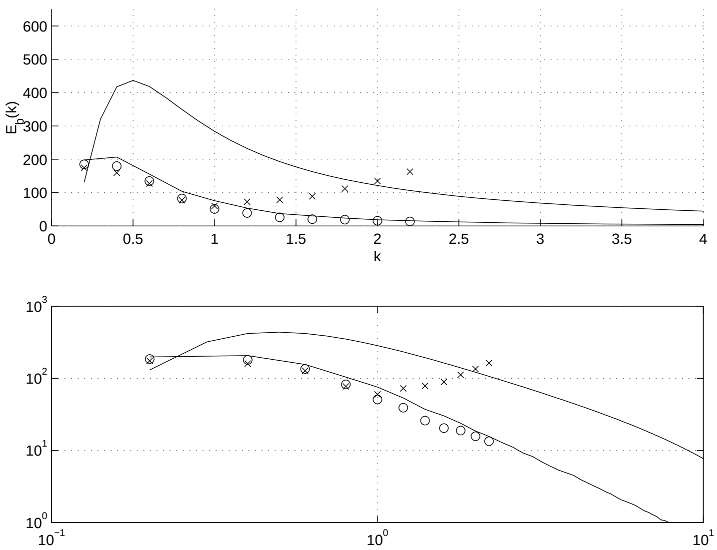

Figure 1 : magnetic spectrum

4 Discussion

The present study has shown that the concept of LES can, at the very least, be used in MHD flows. Of course, the comparison reported here have been done by using relatively small magnetic Reynolds numbers and in a very simple geometry. More complex MHD flows corresponding to systems with a constant magnetic field (which will be useful for exploring anisotropy in turbulent transport) or toroidal geometry will be considered in the future.

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