Modelling RF heating in tokamaks: comments on methods and nomenclature

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Introduction

Radio frequency (RF) heating models are commonly derived using either the Hamiltonian or the trajectory integral approach. The former (action-angle) approach is elegant but the actions are not very transparent variables. The trajectory integral approach is much more intuitive but its practical application becomes cumbersome as well when accounting rigorously for nontrivial geometries. The present paper focuses on two aspects of the self-consistent wave + Fokker-Planck equation model developed by Lamalle and Van Eester (see [Lamalle, 1997&1998] for the wave and [Van Eester, 1995&1998] for the Fokker-Planck part). The first discussion is on the equivalence of the Hamiltonian and the trajectory integral approaches, and yields a formulation that goes beyond tokamak geometry. The second is on the inappropriateness of the term "collisionless" damping in a tokamak: the usual causal "recipe" $\omega \to \omega + i\nu$ with small but positive ν is no longer unambiguous.

Equivalence of evaluation methods and a glance beyond tokamak geometry

Both the Hamiltonian and trajectory integral approaches start from the quasi-linear expressions for the RF perturbed distribution function f_{RF} (rapid time scale; wave equation) and the RF diffusion operator $\boldsymbol{Q}$ (slow time scale; Fokker-Planck equation):

$$f_{RF} = -\int_{-\infty}^{t} dt' \, \frac{q}{m}[\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}}].\nabla_{\bar{\mathbf{v}}} F_o$$

$$\boldsymbol{Q}F_o = \left.\frac{\partial F_o}{\partial t}\right|_{RF} = \frac{1}{2}\mathrm{Re} < \nabla_{\bar{\mathbf{v}}}.\frac{q}{m}[\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}}]^* \int_{-\infty}^{t} dt' \, \frac{q}{m}[\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}}].\nabla_{\bar{\mathbf{v}}} > F_o$$

in which q, m, $\bar{\mathbf{v}}$, F_o, t, $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ are the charge, mass, velocity, slowly varying distribution, and (RF) electric and magnetic fields, respectively.

The Hamiltonian H is the crucial ingredient in the action-angle approach (see [Kaufman, 1972] for the basic action-angle formalism and [Bécoulet et al., 1991], [Carlsson et al., 1997] for applications). Rather than using the position $\bar{\mathbf{x}}$ and its associated momentum, one writes the particle equations of motion in terms of the actions $\bar{\mathbf{J}}$ (which are constants of the motion in the absence of the perturbing RF field) and their associated angles $\bar{\Phi}$: $\dot{\bar{\Phi}} = \frac{\partial H}{\partial \bar{\mathbf{J}}}$ and $\dot{\bar{\mathbf{J}}} = -\frac{\partial H}{\partial \bar{\Phi}}$. The unperturbed motion is then described by $\dot{\bar{\Phi}} = \bar{\omega}(\bar{\mathbf{J}})$ and $\dot{\bar{\mathbf{J}}} = \bar{\mathbf{0}}$, the frequencies $\bar{\omega} = \frac{\partial H_o}{\partial \bar{\mathbf{J}}} = (\omega_b, \omega_d, \omega_g)$ in which characterise the 3 oscillatory aspects of the unperturbed particle motion in a tokamak (bounce, toroidal drift and gyro- motion); H_o is the unperturbed Hamiltonian. Since $\bar{\omega}$ only depends on the actions, the associated angles vary *linearly* with time. The perturbed actions vary with time as prescribed by

$$\delta\dot{\bar{\mathbf{J}}} = -\frac{\partial \delta H}{\partial \bar{\Phi}} = -\frac{\partial[-\frac{q\bar{\mathbf{v}}.\bar{\mathbf{E}}}{i\omega}]}{\partial \bar{\Phi}} = \frac{1}{i\omega}\frac{\partial \dot{\varepsilon}}{\partial \bar{\Phi}} = \sum_{\bar{\mathbf{l}}} \frac{\bar{\mathbf{l}}}{\omega}\dot{\varepsilon}_{\bar{\mathbf{l}}} e^{i[\bar{\mathbf{l}}.\bar{\Phi}-\omega t]},$$

where $\bar{\mathbf{l}}$ is the vector of mode numbers $\bar{\mathbf{l}} = (l_b, l_d, l_g)$ associated with the angles $\bar{\Phi}$, ε is the energy and ω the driver frequency. The notation $\dot{g} = \bar{\mathbf{a}}_{RF}.\nabla_{\bar{\mathbf{v}}} g = \frac{q}{m}[\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}}].\nabla_{\bar{\mathbf{v}}} g$ is adopted for the (velocity related part of the) RF induced time derivative. The RF induced energy change is e.g. $\dot{\varepsilon} = q\bar{\mathbf{E}}.\bar{\mathbf{v}}$.

The <u>trajectory integral approach</u> adopts a set of constants of motion (c.o.m.) $\bar{\Lambda}$ to simplify the quasi-linear description. The perturbed distribution function can then e.g. be written

$$f_{RF} = -\frac{\partial F_o(\bar{\Lambda})}{\partial \bar{\Lambda}}.\int_{-\infty}^{t} dt' \,\dot{\bar{\Lambda}},$$

the integrand in which only contains $\dot{\Lambda}_k$, which can be related to $\dot{\varepsilon} = q\bar{\mathbf{E}}.\bar{\mathbf{v}}$ using [Landau & Lifshitz, 1960]

$$\bar{\mathbf{a}}_{RF} = \frac{q}{m}[\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}}] = \frac{i}{\omega m}[\frac{d}{dt}\nabla_{\bar{\mathbf{v}}} - \nabla_{\bar{\mathbf{x}}}]\dot{\varepsilon}. \tag{1}$$

The above allows to write $\dot{\Lambda}_k = \bar{\mathbf{a}}_{RF}.\nabla_{\bar{\mathbf{v}}}\Lambda_k = \frac{i}{\omega m}[\frac{d}{dt}(\nabla_{\bar{\mathbf{v}}}\Lambda_k.\nabla_{\bar{\mathbf{v}}}\dot{\varepsilon}) - \boldsymbol{D}_{\Lambda_k}\dot{\varepsilon}\,]$ and so the perturbed distribution function is the sum of 2 terms,

$$f_{RF} = \sum_k \frac{i}{\omega m}\frac{\partial F_o(\bar{\Lambda})}{\partial \Lambda_k}[\,-\nabla_{\bar{\mathbf{v}}}\Lambda_k.\nabla_{\bar{\mathbf{v}}}\dot{\varepsilon} + \int_{-\infty}^{t} dt' \boldsymbol{D}_{\Lambda_k}\dot{\varepsilon}\,], \tag{2}$$

the first of which is local and purely reactive; the operator $\boldsymbol{D}$ is given by $\boldsymbol{D}_g = \frac{d}{dt}[\nabla_{\bar{\mathbf{v}}} g].\nabla_{\bar{\mathbf{v}}} + \nabla_{\bar{\mathbf{v}}} g.\nabla_{\bar{\mathbf{x}}}$. Choosing c.o.m. for which $\boldsymbol{D}$ is simple allows to proceed further analytically. The energy ε and the magnetic moment μ are 2 natural c.o.m. in magnetic configurations. In the axisymmetric tokamak, the toroidal angular momentum p_φ is a third one. One readily observes that $\boldsymbol{D}_\varepsilon = m[\frac{d}{dt} - \frac{\partial}{\partial t}]$. Lamalle further found that $\boldsymbol{D}_\mu = -q\frac{\partial}{\partial \phi}$ and $\boldsymbol{D}_{p_\varphi} = m\frac{\partial}{\partial \varphi}$ [Lamalle, 1997], ϕ and φ being the usual gyro- and toroidal angles. Through generating functions relating sets of canonical variables (see e.g. [Tolman, 1938]), one more generally finds that [Van Eester, 1999]

$$\boldsymbol{D}_{J_k} = m\frac{\partial}{\partial \Phi_k} \tag{3}$$

holds in any geometry when the actions $\bar{\mathbf{J}}$ and their associated angles $\bar{\Phi}$ are used as independent variables.

Using the simplest possible decorrelation scheme ($\omega \rightarrow \omega + i\nu$) and action-angle variables, *either* of both approaches yield the following expressions:

$$f_{RF} = -\frac{\partial F_o}{\partial \bar{\Lambda}}.\frac{i}{m\omega}\nabla_{\bar{\mathbf{v}}}\dot{\varepsilon}.\nabla_{\bar{\mathbf{v}}}\bar{\Lambda} + \frac{\partial F_o}{\partial \bar{\Lambda}}.\frac{i}{\omega}\sum_{\bar{\mathbf{l}}}\frac{\bar{\mathbf{h}}_{\bar{\mathbf{l}}}\,\dot{\varepsilon}_{\bar{\mathbf{l}}} e^{i(\bar{\mathbf{l}}.\bar{\Phi}-\omega t)}}{\bar{\mathbf{l}}.\bar{\omega} - \omega}$$

$$\boldsymbol{Q} = \sum_{\bar{\mathbf{l}}} \frac{1}{\mathsf{J}} \frac{\partial}{\partial \bar{\Lambda}} . \bar{h}_{\bar{\mathbf{l}}} \quad \mathsf{J} \frac{|\dot{\varepsilon}_{\bar{\mathbf{l}}}|^2}{\omega^2} \frac{1}{2} \mathrm{Re}\{\frac{1}{i[\bar{\mathbf{l}}.\bar{\omega} - \omega]}\} \bar{\mathbf{h}}_{\bar{\mathbf{l}}} \quad . \frac{\partial}{\partial \bar{\Lambda}},$$

the vector $\bar{\mathbf{h}}_{\bar{\mathbf{l}}}$ in which is $\bar{\mathbf{l}} = (l_b, l_d, l_g)$; J is the Jacobian. Adopting $\frac{\varepsilon}{\omega}$ instead of the action associated with the bounce motion as one of the c.o.m. (the former is intuitive and easy to compute while the latter is neither) amounts to substituting 1 for m_b in $\bar{\mathbf{h}}_{\bar{\mathbf{l}}}$ and adding the purely reactive local term $\frac{i\dot{\varepsilon}}{\omega}\frac{\partial F_o}{\partial \varepsilon}$ to f_{RF}. Since only poloidal coupling occurs, either the usual or the Hamiltonian toroidal and gyro- angles can be adopted: $\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \Phi_d}$ and $\frac{\partial}{\partial \phi} = \frac{\partial}{\partial \Phi_g}$.

The demonstration of the equivalence of both approaches hinges on Eq.(1) and on the operator $\boldsymbol{D}$. They equally allow to go beyond tokamak geometry: it suffices to replace *(p_φ ,φ)* by the new action and its associated angle.

Practical expressions

Evaluating f_{RF} and $\boldsymbol{Q}$ is trivial once the complex $\bar{\Phi}$-Fourier amplitudes of $q\bar{\mathbf{E}}.\bar{\mathbf{v}}$ are found. This is immediate for the cyclotron and toroidal modes (orthogonality: $m_d = n$, $m_g = -N$, $\dot{\phi} = -\Omega$ with Ω the cyclotron frequency) but due to poloidal coupling it is not straightforward at all for the bounce spectrum: bounce modes and poloidal modes can in general not be confused. In case the phase Θ of $q\bar{\mathbf{E}}.\bar{\mathbf{v}}$ is rapidly varying, the stationary phase method can be used; the stationary phase condition is $k_{//}v_{//} + N\Omega = \bar{\mathbf{l}}.\bar{\omega}$. The sum on the bounce modes then becomes a bounce integral. Provided the poloidal modes m and m' are not too far apart, the variational form of the active RF term in the wave equation is

$$... q\bar{\mathbf{F}}^*.\bar{\mathbf{v}}\int_{-\infty}^{t} dt' q\bar{\mathbf{E}}.\bar{\mathbf{v}} = ... q^2 \sum_N \sum_n \sum_m \sum_{m'} \frac{1}{2\pi}\int_0^{2\pi} d\Phi_b \frac{\mathsf{L}^*(\bar{\mathbf{F}}_{m'})\mathsf{L}(\bar{\mathbf{E}}_m) e^{i[m-m']\theta(\Phi_b)}}{i[k_{//,(m+m')/2} v_{//} + N\Omega - \omega]}$$

in which L is the (generalised) Kennel&Engelmann operator, $k_{//}$ is the parallel wave number and $v_{//}$ the parallel velocity. Substituting $\bar{\mathbf{E}}$ for the test function $\bar{\mathbf{F}}$, this term also appears in the Fokker-Planck equation. The corresponding expression for the absorbed power is just the bounce averaged version of its uniform plasma counterpart.

"Collisionless" damping

Finite damping only occurs if decorrelation is present. In a uniform plasma the absorbed power is of the form

$$P_{abs} = \frac{1}{2}\mathrm{Re}\{\int d\bar{\mathbf{x}}\bar{\mathbf{E}}^*.\bar{\mathbf{J}}\} = \frac{1}{2}\mathrm{Re}\{...\int d\bar{\mathbf{v}} d\bar{\mathbf{x}} \frac{...}{i[k_{//}v_{//} + N\Omega - \omega]}\} \qquad (4)$$

and one commonly relies on causality to interpret ω as $\omega + i\nu$ while taking the decorrelation frequency ν *vanishingly small* and positive. Since collisionality can be accounted for in the dielectric response by identifying a finite ν with the collision frequency, since $\nu << \omega$ for this choice and since it makes no difference in practice if ν is taken very small or infinitesimal ($\nu \to 0^+$), the term "collisionless" damping is justified. Absorption takes place when $k_{//}v_{//} + n\Omega - \omega = 0$. In a tokamak, the absorbed power is given by

$$P_{abs} = \frac{1}{2}\mathrm{Re}\{\int d\bar{\mathbf{x}}\bar{\mathbf{E}}^*.\bar{\mathbf{J}}\} = \frac{1}{2}\mathrm{Re}\{...\int d\bar{\Lambda} \frac{...}{i[\bar{\mathbf{l}}.\bar{\omega} - \omega]}\} \qquad (5)$$

closely related to Eq.(4); for the particular case of well passing particles the relations $\omega_b \approx v_{//} \sin\Theta / \left|\partial \bar{\mathbf{x}} / \partial\theta\right|$ and $\omega_d \approx v_{//} \cos\Theta / R$ hold, and the bounce and poloidal mode numbers coincide. There is, however, one important difference: in the uniform case the total absorption is an *integral* over the continuous wave spectrum, while in the tokamak it is a discrete sum. The consequence of the $k_{//}$-quantization is that the condition $\nu << \omega$ no longer uniquely defines the "collisionless" regime. The bounce frequency being small w.r.t. ω as well, three different regimes now occur: When the bounce frequency is small w.r.t. the decorrelation frequency, the function

$$\mathrm{Re}\{\frac{1}{i(\bar{\mathbf{l}}.\bar{\omega} - [\omega + i\nu])}\} = \frac{1}{\omega_b} \frac{\frac{\nu}{\omega_b}}{\left(l_b - l_{b_{res}}\right)^2 + \left(\frac{\nu}{\omega_b}\right)^2} \text{ with } l_{b_{res}} = \frac{\omega - l_d\omega_d - l_g\omega_g}{\omega_b}$$

varies little when going from l_b to $l_b + 1$. As a consequence, the sum over the bounce spectrum yields essentially the same result as the integral in which l_b is interpreted as a continuous variable. Note, however, that ν is necessarily nonzero (i.e. the regime is essentially *collisional* when thinking of ν as a collision frequency) since it is large compared to ω_b. When on the contrary the bounce frequency is large compared to ν, the same reasoning yields a regime much more entitled to the label "collisionless", but only in the exceptional cases when the above defined $l_{b_{res}}$ is an integer, the absorption is nonzero. This so-called superadiabatic regime is, however, not what is commonly known as "collisionless ". Rather than being localized at the poloidal position satisfying the resonance condition $k_{//}v_{//} + N\Omega - \omega = 0$, this damping occurs globally on a finite set of magnetic surfaces. A third regime occurs when ω_b and ν are comparable. Constructive or destructive interference of various encounters of a particle with the same resonance is then important. This is the only regime where the details of the decorrelation model actually matter (see [Van Eester et al., 1996]).

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