Transition from time-continuous systems to discrete mappings

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Detailed transport studies in plasmas require the solution of the time evolution of many different initial positions of test particles in the phase space of the systems to be investigated. To reduce this amount of numerical work, one would like to replace the integration of the time-continues system with a mapping [3]. Classic examples for such approaches are the standard map and the Tokamap (Balescu *et al* [1]). These mappings are derived from so-called kicked systems, like the kicked rotator. This basically means, that the system consists of an integrable part and a non-integrable perturbation. The perturbation is only present at fixed singular points in time and zero for all other times. Thus the system remains integrable for all times except the ones, when the perturbation is present. The solution for these singular points is usually found by approaching the point from either later or earlier times ($t \pm \varepsilon$) and calculating the limit for $\varepsilon \to 0$. However this procedure does not necessarily yield the correct result. This can be seen by comparing the results from a map with the numerical calculation of the given system with an approximative δ-function. This is shown for the example of the Tokamap. A new procedure to evaluate the system at these singular points is shown (a so-called symmetric evaluation of the δ-function) and the near perfect agreement with the result of the numerical calculation is demonstrated. It is further possible to use a non-singular time dependence for the perturbation in the numeric calculation and thus the effect of approximating a timely distributed perturbation with a δ-function can be investigated.

The example Hamiltonian

For the derivation of the Tokamap [1] the following Hamiltonian was used by Balescu et al.:

$$H = \int^{\psi} \mathrm{d}\psi' \;\; W(\psi') - \frac{K}{(2\pi)^2} \cdot \frac{A \cdot \psi}{1 + A \cdot \psi} \cdot \cos(2\pi \cdot \theta) \cdot f(t)$$

This Hamiltonian, which describes the motion of magnetic field lines in a Tokamak, is quite often used as a simple approximation for the real magnetic field found in the experiments. It describes a system with a given winding number $W(\psi)$ (integrable part) and a perturbation (non-linear and non-integrable), that will vanish for $\psi = 0$. This specific form of the perturbation was introduced by Balescu et al. to avoid the unphysical case with $\psi < 0$, which occurs in the standard map.

Thus we obtain the equations of motion for the field lines:

$$\begin{aligned}\dot{\psi} &= -\frac{\partial H}{\partial \theta} = -\frac{K}{2\pi}\cdot\frac{A\cdot\psi}{1+A\cdot\psi}\cdot\sin(2\pi\cdot\theta)\cdot f(t)\\ \dot{\theta} &= \frac{\partial H}{\partial \psi} = W(\psi)-\frac{K}{(2\pi)^2}\cdot\frac{A}{(1+A\cdot\psi)^2}\cdot\cos(2\pi\cdot\theta)\cdot f(t)\end{aligned}$$

Tokamap

Since the generic case with a smooth and distributed function $f(t)$ cannot be integrated analytically, we have to choose this function as a delta function.

$$f(t)=\sum_{k=-\infty}^{\infty}\delta(k-t)$$

Doing so, we obtain the Tokamap.

$$\begin{aligned}\psi_{n+1} &= \psi_n-\frac{K}{2\pi}\cdot\frac{A\cdot\psi_{n+1}}{1+A\cdot\psi_{n+1}}\cdot\sin(2\pi\cdot\theta_n)\\ \theta_{n+1} &= \theta_n+W(\psi_{n+1})-\frac{K}{(2\pi)^2}\cdot\frac{A}{(1+A\cdot\psi_{n+1})^2}\cdot\cos(2\pi\cdot\theta_n)\end{aligned}$$

This implicit form can be rewritten in an explicit form:

$$P(\psi_n,\theta_n)=1-A\cdot\psi_n+\frac{K}{2\pi}\cdot A\cdot\sin(2\pi\cdot\theta_n)$$

$$\begin{aligned}\psi_{n+1} &= \frac{1}{2\cdot A}\left\{\sqrt{P(\psi_n,\theta_n)^2+4\cdot A\cdot\psi_n}-P(\psi_n,\theta_n)\right\}\\ \theta_{n+1} &= \theta_n+W(\psi_{n+1})-\frac{K}{(2\pi)^2}\cdot\frac{A}{(1+A\cdot\psi_{n+1})^2}\cdot\cos(2\pi\cdot\theta_n)\end{aligned}$$

The winding number can be chosen from a wide range of possibilities. We use here a realistic one for TEXTOR-94.

$$W(\psi)=\frac{1}{1+d\cdot\psi}\quad\text{with}\quad d=3$$

Numerical integration

To estimate the effect of choosing $f(t)$ to be a delta function, we compare the results for the Tokamap with the result of a numerical calculation of the Hamiltonian with a symplectic integrator [2, 4]. The delta function is approximated by a Gaussian function.

$$f(t,a)=\sum_{k=-\infty}^{\infty}\frac{a}{\sqrt{\pi}}\cdot\exp\left\{-a^2\cdot(t-k)^2\right\}$$

Symmetric map

Comparing the results from the Tokamap and the integration shows a huge difference. When looking at the procedure, the Tokamap was derived from the Hamiltonian system, one finds out, that the difference comes from the non-symmetric evaluation of the delta function. So it is quite obvious to try a different approach with a symmetric evaluation of the delta function.
Starting with the same Hamiltonian

$$H = \int^{\psi} \mathrm{d}\psi' \;\; W(\psi') + f(\psi) \cdot g(\theta) \cdot \sum_{k=-\infty}^{\infty} \delta(t-k)$$

$$\text{with} \quad f(\psi) = -\frac{K}{(2\pi)^2} \cdot \frac{A \cdot \psi}{1 + A \cdot \psi} \quad \text{and} \quad g(\theta) = \cos(2\pi \cdot \theta)$$

leads to the equations of motion:

$$\begin{aligned} \dot{\psi} &= -\frac{\partial H}{\partial \theta} = \qquad\qquad - \; f(\psi) \cdot \frac{\partial g(\theta)}{\partial \theta} \cdot \sum_{k=-\infty}^{\infty} \delta(t-k) \\ \dot{\theta} &= \frac{\partial H}{\partial \psi} = W(\psi) + \frac{\partial f(\psi)}{\partial \psi} \cdot g(\theta) \cdot \sum_{k=-\infty}^{\infty} \delta(t-k) \end{aligned}$$

Now we calculate the step from time t_n to t_n^+.

$$\psi_n^+ = \psi_n - \frac{1}{2} \cdot f(\psi_n) \cdot \frac{\partial g(\theta_n)}{\partial \theta} , \quad \theta_n^+ = \theta_n + \frac{1}{2} \cdot \frac{\partial f(\psi_n)}{\partial \psi} \cdot g(\theta_n)$$

Then continue from t_n^+ to t_{n+1}^-.

$$\psi_{n+1}^- = \psi_n^+ , \quad \theta_{n+1}^- = \theta_n^+ + W(\psi_n^+)$$

And finally from t_{n+1}^- to t_{n+1}.

$$\begin{aligned} \psi_{n+1} &= \psi_{n+1}^- - \frac{1}{2} \cdot f(\psi_{n+1}) \cdot \frac{\partial g(\theta_{n+1})}{\partial \theta} \\ \theta_{n+1} &= \theta_{n+1}^- + \frac{1}{2} \cdot \frac{\partial f(\psi_{n+1})}{\partial \psi} \cdot g(\theta_{n+1}) \end{aligned}$$

Putting everything together leads to the final form of the symmetric map:

$$\begin{aligned} \psi_{n+1} &= \psi_n - \frac{1}{2} \cdot f(\psi_n) \cdot \frac{\partial g(\theta_n)}{\partial \theta} - \frac{1}{2} \cdot f(\psi_{n+1}) \cdot \frac{\partial g(\theta_{n+1})}{\partial \theta} \\ \theta_{n+1} &= \theta_n + W\left(\psi_n - \frac{1}{2} \cdot f(\psi_n) \cdot \frac{\partial g(\theta_n)}{\partial \theta}\right) + \cdots \\ &\qquad \cdots + \frac{1}{2} \cdot \frac{\partial f(\psi_n)}{\partial \psi} \cdot g(\theta_n) + \frac{1}{2} \cdot \frac{\partial f(\psi_{n+1})}{\partial \psi} \cdot g(\theta_{n+1}) \end{aligned}$$

Conclusions

The comparison of these three approaches shows clearly, that the correct evaluation of the δ-function is the most important point. The more physically motivated approach is in very well accordance with the numerical integration, while the more mathematical derivation of the Tokamap yields a quite different result. This fact must be taken into account, if one wants to construct a map for a given Hamiltonian system.

Further numerical comparisons for different values of a, the sharpness of the approximated δ-function, shows a rather small difference. Thus the approximation of a broader distributed perturbation with a sharp δ-function is quite well. This means, that symmetric maps can indeed provide a fast method to study the properties of these kinds of Hamiltonian systems.

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References

[1] R. Balescu, M. Vlad and F. Spineanu, *Tokamap: A Hamiltonian twist map for magnetic field lines in a toroidal geometry*, Phys. Rev. E **58** (1998) 951–964

[2] Etienne Forest and Ronald D. Ruth, *Fourth-order symplectic integration*, Physica D **43** (1990) 105–117

[3] K. H. Spatschek, M. Eberhard and H. Friedel, *On models for magnetic field line diffusion*, Physicalia Mag. **20** (1998) 85–93

[4] Haruo Yoshida, *Construction of higher order symplectic integrators*, Phys. Lett. A **150** (1990) 262–268