# Finite time singularities and regular motion in magnetic reconnection

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**1.  Hamiltonian reconnection**    Reconnection of magnetic fields in plasmas is thought to play a role at small scales, e.g. in fluctuations and in the formation of current filaments, as well as in large scale phenomena in astrophysics and disruptive relaxations of tokamak plasmas. In sufficiently well conducting plasmas, large reconnection rates are attributed not to resistivity but to collisionless reconnection in a layer of width $d_e = c/\omega_{pe}$ (the electron inertial skin depth). Neglecting dissipation, the process can be described by a Hamiltonian system.

The present paper considers two-dimensional (2D) systems described by reduced two-fluid equations that include the effects of parallel compressibility of the electrons, introducing an additional length scale $\rho_s = (T_e/T_i)^{1/2}\rho_i$ (ion gyroradius at the electron temperature). This three field model is Hamiltonian [1].

It is known that, once the nonlinear stage of Hamiltonian reconnection is reached, the width of the current layer in the reconnection zone can continue to shrink indefinitely [2, 3]. Numerical simulations show that for $\rho_s > 0$ scale collapses faster than exponential occur [4], giving rise to the question whether such systems exhibit finite time singularities or remain regular indefinitely.

The question of finite-time blow-up (f.t.b.) is particularly important because the model has three infinite series of conserved quantities [8]. In the event of f.t.b. the conservation laws of the system may be valid only up to that time.

**2.  Finite time singularities**    The possibility of f.t.b. in continuous systems has been investigated extensively. In inviscid incompressible fluid flow, described by Euler's equations, numerical evidence for f.t.b. exists, involving complex 3D flows. In fact, 2D Euler flow has long been known to be regular at all times. Numerical and analytical evidence indicates that this situation is parallelled in ideal MHD: 2D systems are regular at all times and 3D systems only up to a finite time. In contrast, the electron-MHD model admits f.t.b. even in 2D motion [5]. All these systems have an infinity of conserved quantities: vortex lines are frozen in the velocity field in Eulerian flow. In MHD magnetic field lines instead of vortex lines are frozen-in. All these properties are guaranteed only up to the time of a singularity.

The present paper is restricted to 2-dimensional systems. The reduced two-fluid model, depending on the values of $d_e$, $\rho_s$, and $\rho_i$, can be proven to either have no f.t.b., as in the 2-D Euler case, or have f.t.b. which, as in the case of electron-MHD, can be given analytically. This clear-cut situation contrasts the case of reduced ideal MHD where strong evidence, but no proof, exists for the impossibility of f.t.b.

In the limit $d_e \to 0$, the $\rho_s$-contribution is known to make finite-time singularities possible, in which e.g. the current layer width shrinks indefinitely.

Finite time singularities are shown to be absent for $d_e > 0$. However, if in addition some finite ion temperature effects are included in the model (preserving its Hamiltonian structure), that proof does not hold and finite-time blow-up cannot be excluded.

**3. Three field model** The model considered here is based on the extension of reduced MHD to the two-fluid model given in Ref. [6] (a more general model was used in [7]), where the contribution of electron inertia to the generalized Ohm's law provides the mechanism for collisionless reconnection. Also the electron gyroviscosity and the Hall term are retained in the generalized Ohm's law. For the electrons the isothermal equation of state is adopted.

The reduced equations are based on a magnetic field with a dominant, constant, component in the $z$-direction, $\mathbf{B} = B_0(\mathbf{e}_z + \mathbf{e}_z \times \nabla\psi)$. The electric field is $\mathbf{E} = B_0(-\nabla\phi + \mathbf{e}_z\partial_t\psi)$, where $\psi$ and $\phi$ are the normalized flux and scalar potential. We have $|\nabla\psi| \ll 1$, and $\partial_z = 0$. We define $j$ and $\omega$ as the current density and vorticity in the $z$-direction, and $n$ as the density. The following set of equations is derived,

$$D_t(\psi - d_e^2 j) = \rho_s^2[N, \psi], \tag{1a}$$

$$D_t N = v_A^2[\psi, j], \tag{1b}$$

$$D_t(N - \rho_i^2\Delta N - \omega) = 0, \tag{1c}$$

where

$$j = \Delta\psi, \qquad \omega = \Delta\phi, \qquad N = \Omega_i \ln(n/n_0),$$

and where $D_t = \partial_t + [\phi, \cdots]$, $[A, B] = \partial_x A\,\partial_y B - \partial_y A\,\partial_x B$. Equation (1c) represents the ion dynamics in a simplified way and necessarily has a very limited validity. However, Eq. (1c) correctly represents the cold ion limit ($\rho_i \to 0$) as well as the large-$\rho_i$ limit, where the Boltzmann response of the ions $n \sim \exp(-\phi/T_i)$ is a solution of Eq. (1c). Compared to Ref. [6] we have neglected a term $\rho_i^2(\nabla\bullet[\phi, \nabla N] - [\phi, \Delta N])$. Retaining this term would complicate the expressions without changing our qualitative results.

The above three field model can be written in the form [8]

$$\partial_t\omega_j + [\phi_j, \omega_j] = 0, \qquad j = +, -, 3, \tag{2}$$

with 3 conserved generalized vorticities

$$\omega_\pm = N \pm (v_A/\rho_s d_e)(\psi - d_e^2 j), \tag{3a}$$

$$\omega_3 = N - \omega - \rho_i^2\Delta N, \tag{3b}$$

advected by the incompressible flows defined by the stream functions

$$\phi_\pm = \phi \pm d_e^{-1}\rho_s v_A\psi, \tag{4a}$$

$$\phi_3 = \phi. \tag{4b}$$

Equations (2) imply the existence of three infinite series of conserved integrals $\int d^2x(\omega_j)^n$. The full system consists of Eqs. (2) supplemented by relations between the fluxes $\omega_j$ and the stream functions $\phi_j$,

$$\phi_j(\mathbf{x}) = \int d^2x'\, G_{jk}(|\mathbf{x} - \mathbf{x}'|)\, \omega_k(\mathbf{x}'), \tag{5}$$

where the Green's functions are derived from Eqs. (3,4),

$$G_{jk}(x) = \alpha_{jk}\frac{\ln(x)}{2\pi} - \beta_{jk}\frac{K_0(x/d_e)}{2\pi} - \gamma_{jk}\rho_i^2\delta(x), \tag{6}$$

where

$$\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{7}$$

The Bessel function $K_0(x)$ behaves like $-\ln(x)$ for $x \to 0$. Therefore the terms depending on $\alpha_{jk}$ and $\beta_{jk}$ have the same logarithmic singularity as the Green's function in the Euler case. However, the third term gives rise to a $\delta$-function singularity unless $\rho_i = 0$.

As an application of these expressions, one can consider $\delta$-function solutions for $\omega_j$ in Eqs. (2). Then, the Green's functions (6) provide us with a generalized point-vortex model [8, 9] analogous to the Kirchhoff equations for 2D Euler point vortices,

$$\frac{d\mathbf{x}_m}{dt} = -\nabla \phi_m \times \mathbf{e}_z, \qquad \phi_m(\mathbf{x}_m) = \sum_n \kappa_n G_{j_m j_n}(|\mathbf{x}_m - \mathbf{x}_n|).$$

Here $\mathbf{x}_m$, $\kappa_n$, and $j_m$ are the position, strength, and type $(+, -, 3)$ of the $m$-th vortex. Note that the point-vortex model is well-defined only for $\rho_i = 0$, because the $G_{jk}$ should not contain $\delta$-function singularities.

**4.  Regularity of 2D Euler flow**    Before addressing the regularity of the above system, we note that it is a generalization of the 2D Euler system, given by

$$\partial_t \omega + [\phi, \omega] = 0, \qquad \omega = \Delta \phi.$$

The potential can be expressed in terms of the vorticity as $\phi(\mathbf{x}) = \int d^2x' \, G(\mathbf{x}, \mathbf{x}') \, \omega(\mathbf{x}')$ where, in the case of an infinite domain, $G(\mathbf{x}, \mathbf{x}') = -(2\pi)^{-1} \ln |\mathbf{x} - \mathbf{x}'|$. The form of $G$ is different for other (e.g. double-periodic) domains, but the singularity remains logarithmic. Specifically: for every bounded domain $D$ constants $c_1$, $c_2$ exist such that for all $\mathbf{x}, \mathbf{y} \in D$,

$$|\nabla G(\mathbf{x}, \mathbf{y})| \le c_1 |\mathbf{x} - \mathbf{y}|^{-1}, \tag{8a}$$

$$|\nabla \nabla G(\mathbf{x}, \mathbf{y})| \le c_2 |\mathbf{x} - \mathbf{y}|^{-2}. \tag{8b}$$

From these inequalities one can derive that for some constant $c_3$,

$$\int_D d^2z \left| \nabla_z G(\mathbf{x}, \mathbf{z}) - \nabla_z G(\mathbf{y}, \mathbf{z}) \right| \le c_3 |\mathbf{x} - \mathbf{y}| \ln \frac{e|D|}{|\mathbf{x} - \mathbf{y}|}, \tag{9}$$

where $|D|$ is the diameter of $D$. Several conservation properties can be derived from the fact that $\omega$ is advected by the flow. We merely need the fact that

$$\Omega \equiv \sup_x |\omega(\mathbf{x})| = \text{constant}. \tag{10}$$

This set of properties of the 2D Euler system suffices to prove (see e.g. the review by Rose and Sulem [10]) the theorem: **If the velocity is n times (n $\ge$ 0) continuously differentiable or C$^\infty$ on a bounded domain, it will remain so indefinitely.** This result guarantees that smooth initial conditions cannot lead to finite-time blow-up.

**5.  Regularity in the three-field model**    The main result of this paper is to extend the above theorem to the three-field model. This can be achieved following the proof of Ref. [10] by showing that the obvious extensions of Eqs. (8) and (10) to three vorticities $\omega_j$ and to the Green's functions (6) holds. Four cases need to be considered.

**Case i: d$_e$ = 0, T$_i$ > 0.** If electron inertia is neglected, $d_e = 0$, the structure with 3 conserved fields $\omega_j$ breaks down because in this case $\omega_\pm$ and $\phi_\pm$ cannot be defined by Eqs. (3a,4a). Near singularity, with $n \sim \exp(-\phi/T_i)$, the system reduces to $\partial_t \psi + (\rho_i^2 + \rho_s^2)[\phi, \psi] = 0$, $\partial_t \phi + v_A^2 \rho_i^{-2}[\psi, j] = 0$, for which solutions with f.t.b. are known [11].

**Case ii: d$_e$ = 0, T$_i$ = 0.** Near a singularity Eqs. (1a,b) reduce to $\partial_t \psi = \rho_s^2[\omega, \psi]$, $\partial_t \omega = v_A^2[\psi, j]$, with singular solutions of the form

$$\psi = \tfrac{1}{12}\rho_s v_A^{-3} \frac{x^3}{(t_0 - t)^3} + \rho_s^{-3} v_A \frac{xy^2}{t_0 - t} + O(r^4), \qquad \omega = -\rho_s^{-2} \frac{xy}{t_0 - t} + O(r^3),$$

for $t \to t_0$. This solution is homologous to the f.t.b. found in 2D electron-MHD [5] and in the above case. Note that the scale-length collapse naturally brings the system into a regime where $d_e$ is no longer negligible, thus resolving the singularity.

**Case *iii*: d$_e$ > 0, T$_i$ > 0.** In the case of hot ions ($\rho_i > 0$) the Green's functions in Eq. (6) contain $\delta$-functions so that the regularity proof does not apply. Indeed, solutions can be constructed which exhibit f.t.b. Near singularity, with $N = -\rho_i^2 \phi$, Eqs. (1a,b) reduce to $\partial_t j + [\phi, j] = 0$ and $\partial_t \phi = -\rho_i^2 v_A^2 [\psi, j]$, with singular solutions such as

$$\phi = \frac{xy}{t_0 - t} + O(r^3), \qquad \psi = \frac{xy^2}{2\rho_i v_A (t - t_0)} + O(r^4). \tag{11}$$

Within our model there is no further scale length (such as $d_e$ in case *(i)*) to resolve the singularity. The f.t.b. is directly related to the existence of the Boltzmann solution $n \sim \exp(-\phi/T_i)$ at $x \ll \rho_i$.

**Case *iv*: d$_e$ > 0, T$_i$ = 0.** This is the only case where three conserved fields $\omega_j$ exist, while the functions $G_{jk}$ contain only logarithmic singularities. In this case the proof sketched above applies and solutions remain regular at all times.

**6.   Summary and discussion**   We have considered a three-field model with $d_e > 0$ in which the three fields $\omega_\pm$, $\omega_3$ are advected by stream functions $\phi_\pm$, $\phi_3$ respectively (with three corresponding infinite series of conserved integrals). Despite these conservation laws magnetic reconnection is possible. Numerical investigations of this process show that faster than exponential scale collapses can take place for $\rho_s > 0$. The present paper shows that for $\rho_i = 0$, $d_e > 0$ such a collapse cannot give rise to f.t.b.: solutions remain regular at all times. For $\rho_i > 0$ f.t.b. cannot be excluded. However, it should be noted that numerically simulated reconnection shows current peaking and a scale collapse at the X-point, and singular solutions such as the one in Eqs. (11) have an incompatible parity. In the singular solutions obtained here $\psi$ has a cubic stationary point. The singular behaviour is not generic and is not expected to arise in arbitrary configurations.

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