

## Hamiltonian map for guiding centres in a perturbed toroidal magnetic geometry

B. Weyssow<sup>1</sup> and J.H. Misguich<sup>2</sup>

<sup>1</sup> Association Euratom-Etat Belge, Statistical Physics and Plasmas research unit,  
Université Libre de Bruxelles, Campus Plaine, CP231,  
B-1050 Bruxelles, Belgium, E-mail : bweyssow@ulb.ac.be

<sup>2</sup> Association Euratom-CEA sur la fusion, DRFC, CEA Cadarache,  
F-13108, Saint-Paul-lez-Durance Cedex, France. E-mail : misguich@drfc.cad.cea.fr

The transport of field lines or of charged particles in a confining device with a stochastic divertor (typically a DED configuration) can in principle be estimated from simplified representations of the dynamics [1]. For example, a symbolic analysis of the Poincaré representation (in the  $\{\theta, \psi\}$  plane) of the *magnetic field lines* dynamics in a toroidal geometry with braided magnetic surfaces has shown that the field lines radial motion is a continuous time random walk [2]. The generalisation of this study to the guiding centre motion implies the search for canonical guiding centre coordinates and the development of new methods for discretising the Hamiltonian. We also present preliminary results obtained from the discrete stroboscopic section of the guiding centre motion in the standard - fully toroidal - perturbed magnetic field.

### 1. Canonical guiding centre coordinates

The guiding centre position has an evolution equation combining a parallel velocity (along the magnetic field line) and a series of drift contributions. According to White et al. [3] an unimportant modification of the drift terms permits to obtain the compact expression  $d\vec{Y}/dt = (U/B) [\vec{B} + \vec{\nabla} \times \rho_{||} \vec{B}] / [1 + \rho_{||} \vec{b} \cdot \vec{\nabla} \times \vec{b}]$  where  $\rho_{||} = U/\Omega$  and  $\Omega = eB/mc$  (the usual guiding centre parameter  $\varepsilon = V_{th}/a\Omega_0$  is introduced below). Here the *pseudo-canonical* coordinates are  $\{\rho, \theta, \zeta, E\}$  where the first three can be identified with the usual toroidal coordinates and the last one with the energy (*i.e.* the numerical value of the Hamiltonian). By a detailed general calculation, we explicitly derived a set of *canonical coordinates*  $\{P_\theta, \theta, P_\zeta, \zeta\}$  by introducing a Clebsch representation for the magnetic field  $\vec{B} = \vec{\nabla} \alpha \times \vec{\nabla} \beta$ . In the particular case of the standard model (Knorr [4]) for the unperturbed part of the magnetic field  $\vec{B} = (1/h) \{ \vec{e}_\zeta + \varepsilon_T (\rho/q(\rho)) \vec{e}_\theta \}$  where  $\varepsilon_T = a/R_0$  (inverse aspect ratio),  $h = 1 + \varepsilon_T \rho \cos \theta$  and  $q(\rho)$  is the safety factor profile, explicit expressions for the canonical momenta are shown to be given by  $P_\zeta = \rho_{||} - \varepsilon^{-1} \int_0^\rho d\rho G(\rho, \theta)$  and  $P_\theta = \varepsilon_T \left[ G \rho \rho_{||} / h + \varepsilon^{-1} \int_0^\rho d\rho \frac{\rho}{h(\rho, \theta)} \right]$ . The function  $G(\rho, \theta) = \varepsilon_T \rho (1/q + \tilde{c} \cos 2\pi\theta)$  depends

on the magnetic field perturbation whose amplitude is chosen to be  $\tilde{c} = -k/(1 + \rho^2/2)^2$  with  $k$  a real number, in order to reproduce the Tokamak result for magnetic field lines.

The expressions given here accurately define a set of *canonical* guiding centre coordinates for a perturbed magnetic field given by a Clebsch representation. Our results are thus similar to the Meiss & Hazeltine [5] canonical coordinates but our method applies to magnetic fields more general than the standard model.

## 2. Symplectic integration of the equations of motion

To proceed, we need a Hamiltonian map defined with the parameters (safety factor *a.s.o.*) of the continuous Hamiltonian. To this end, we use an unambiguous procedure based on first order symplectic integration. The relation between the map and the integration results from the following arguments. A numerical integration of a set of Hamiltonian differential equations  $dq/dt = f(q, p)$ ,  $dp/dt = g(q, p)$  can be performed by a first order (implicit) symplectic integrator  $q_1 = q_0 + \delta t f(q_0, p_1)$ ,  $p_1 = p_0 + \delta t g(q_0, p_1)$  with a small time increment  $\delta t$ . This method applied to the GC motion in a toroidal geometry gives the following difference equations:  $P_{\zeta_1} = P_{\zeta_0}$  is the toroidal invariant;  $P_{\theta_1} = P_{\theta_0} - 2\pi\delta t (\varepsilon_T^2 \Omega^2 / \varepsilon^2 D)(DR - TS) \sin 2\pi\theta_0$ ;  $\theta_1 = \theta_0 + \delta t (\Omega^2 / \varepsilon D)S$  and a de-coupled equation for the toroidal angle  $\zeta_1 = \zeta_0 + \delta t (\varepsilon_T \Omega^2 / \varepsilon D) \left( \rho_{||} D - G \frac{\rho}{h} S \right)$ . The functions appearing in the map are

$$D = \frac{\rho}{h} (1 + G^2) + \varepsilon \rho_{||} \frac{\partial(G\rho/h)}{\partial\rho}, \quad R = -\rho_{||} I + \varepsilon \frac{(U^2 + MB)\rho}{\Omega^2 h},$$

$$S = \frac{U}{\Omega} G + \varepsilon \frac{(U^2 + MB)}{\Omega^2} \frac{1}{B} \frac{\partial B}{\partial\rho}, \quad T = \varepsilon \rho_{||} G \frac{\rho^2}{h^2} + \int_0^\rho d\rho \frac{\rho^2}{h^2} + k \frac{\rho^2}{h} \left( \frac{\varepsilon_T \rho_{||}}{(1 + \rho^2/2)^2} + G I \right)$$

where  $I = \int_0^\rho d\rho \frac{\rho}{(1 + \rho^2/2)^2}$ . The difference equations given here reproduce all the features of

the toroidal trajectories. In particular it includes the two classes of trajectories, the banana and the passing ones. The equations are obviously implicit and require a root finding procedure

at each step to solve the equation for  $\rho$ :  $P_{\theta_1} = \varepsilon_T G \frac{\rho}{h} P_{\zeta_1} + \frac{\varepsilon_T}{\varepsilon} \left( \int_0^\rho d\rho \frac{\rho}{h} + G \frac{\rho}{h} \int_0^\rho d\rho G(\rho, \theta_0) \right)$

### 3. Maps and comb method for the perturbed Hamiltonian

Usually the building of a map consists in switching on the perturbation stroboscopically (see e.g. [6]). This is performed by multiplying the perturbation by a “comb function” whose parameters are defined as follows: its periodicity is  $T = n \delta t$  and the width and height of the teeth are respectively  $\delta t$  and  $T / \delta t$ . This method is applied in Sec. 4 to the derivation of the Tokamap [2]. A procedure departing from this method consists in switching on the whole Hamiltonian at stroboscopic times, which amounts to apply large time steps.

### 4. Derivation of the Tokamap for magnetic field lines

To describe magnetic field line motion we remove all particle drifts and all variations of the parallel velocity. This is accomplished by taking  $M = 0$  and keeping only the dominant terms in  $\varepsilon_T$  and using  $P_\theta = \varepsilon_T \psi / \varepsilon$ . In order to go from the time dependence of the trajectory to the  $\zeta$ -dependence of the field lines, we divide all the equations of motion by the equation for  $\zeta$ . Applying the comb method, we get  $\theta_1 = \theta_0 + \left\{ q^{-1}(\psi_1) - k(1 + \psi_1)^{-2} \cos 2\pi\theta_0 \right\}$ ,  $\psi_1 = \psi_0 - 2\pi k \psi_1 (1 + \psi_1)^{-1} \sin 2\pi\theta_0$  which is nothing else than the Tokamap [2]. In this particular case, since field lines remain on the (unperturbed) magnetic surfaces where  $q(\psi) = cst.$ , the alternative comb method leads to the same map.

### 5. Preliminary results from the alternative comb method

By applying the alternative comb method of Sec. 3, together with the symplectic integrator of Sec. 2, we have obtained promising results describing the GC dynamics in a perturbed magnetic field with the same magnetic perturbation as in the Tokamap. This amounts to use large time steps. Fig.1 shows the time evolution of the radial position of a GC for two different values of the perturbation amplitude:  $k = 0.1$  and  $k = 0.5$ . It is obvious that for small values of the perturbation amplitude the dynamics remains regular (periodic oscillations of the radius) whereas for large values of the perturbation amplitude the dynamics seems to become stochastic. This behaviour is reminiscent of the behaviour of the “classical” standard map.

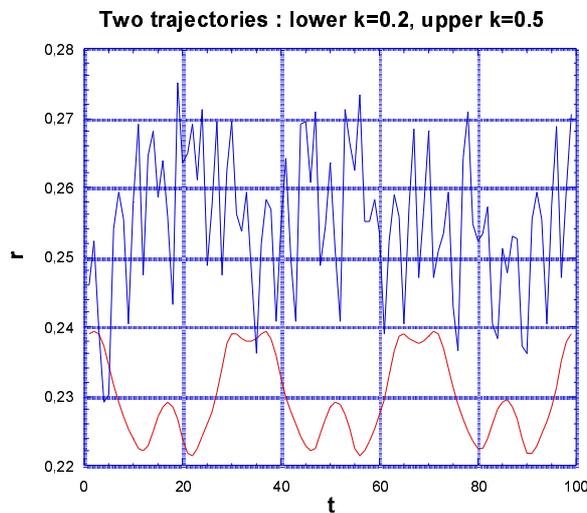


Fig. 1: The discrete equations as a map. The radial position of the GC as a function of time for two values of the perturbation amplitude. Large values of the amplitude lead to chaotic trajectories.

## 6. Conclusions

Starting from the usual GC equations of motion and using a Clebsch representation for the magnetic field, we have been able to derive a set of canonically conjugated coordinates. Explicit results are given for the standard model. A procedure relying on symplectic integration was used to derive a discrete representation of the dynamics. This map can be used in two different ways, either as an integrator of the equations of motion in which case we recover the known classes of trajectories or, as a map in which case the dynamics can become stochastic for sufficiently powerful perturbations. The Tokamap for the magnetic field lines in a toroidal geometry is easily recovered as a limiting case. Preliminary results have been obtained for GC dynamics taking moreover magnetic drifts into account. The analysis of the diffusion process is in progress.

- [1] S.S. Abdullaev, K.H. Finken & K.H. Spatschek, *Phys. Fluids B* **6**,153 (1999)
- [2] R. Balescu, M. Vlad, F. Spineanu, *Phys. Rev. E* **58**, 951 (1998); R. Balescu, *Phys. Rev. E* **58**, 3781 (1998).
- [3] R.B. White, A.H. Boozer & R. Hay, *Phys. Fluids* **25**, 575 (1982).
- [4] G. Knorr, *Phys. Fluids* **8**, 1334 (1965); J. H. Misguich, B. Weyssow & R. Balescu, Report EUR-CEA-FC 1438, Cadarache (1992).
- [5] J.D. Meiss, R.D. Hazeltine, *Phys. Fluids B* **2**, 2563 (1990).
- [6] R.Z. Sagdeev, D.A. Usikov & G.M. Zaslavsky, *Nonlinear Physics*, Harwood Acad. Publ. (1988).