Analytical Solution of Problem of Charge Particle Drift Trajectories in Tokamaks valid for all Plasma Column

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The problem of describing of the drift orbits of charge particles in a tokamak magnetic field is important for both present-day and future devices. Orbit information is required for studying the behaviour of fusion generated α-particles and high-energy ions in tokamaks and for determining the particle and heat transport coefficients in plasma.

The trajectories of circulating particles were first calculated by Budker [1] and those of trapped particles, by Morozov and Solov'ev [2]. Both of this calculations were carried out under assumption that the drift orbits deviate from the magnetic surface only slightly. An attempt to systematically take into account finite drift excursions was first made by Berk and Galeev [3], who derived the corresponding equations but did not investigate them. Particle trajectories that pass through magnetic axis were analysed in detail by many authors (see, e.g., [4-6]). However, even the analysis of this fairly simple cases have not been exhaustive, because the drift equation depends on a large number of parameters and, consequently, is rather complicated.

In analysing particle motion in a tokamak, we assume that the toroidal magnetic field is $B = B_0 / (1 + \varepsilon \cos \theta)$, i.e. that the magnetic field generated by the plasma current is low in comparison with the toroidal field. We also neglect the effect of the electric field on the particle motion.

We use the condition for three quantities - the total energy E, the magnetic moment $\mu = m v_\perp^2 / 2B$, and the longitudinal adiabatic invariant $J = m v_{II} (1 + \varepsilon \cos \theta) - (e / c) \int B_\theta dr$ - to be conserved. Here, v_{II} and $v_\perp$ are the particle velocity components along and across the magnetic field, θ is the poloidal angle, B_θ is the poloidal magnetic field, and $\varepsilon = r / R$ is the inverse aspect ratio; the remaining notations are standard.

To simplify the analysis of the orbit topology, we assume that the safety factor does not change along the drift trajectories.

In this approximation, the expression for the longitudinal invariant yields

$$r^2 = \pm \varsigma \left| \frac{v_{II}}{v} \right| R^2 (1 + \varepsilon \cos \theta) + C , \qquad (1)$$

where $\varsigma = 2\rho q / R$, the plus and minus signs refer to the particles that move along the magnetic field lines and in the opposite direction, ρ is the Larmor radius of a particle, v is the total particle velocity, q is the safety factor, and the constant C is determined by the position of the point on the trajectory. The longitudinal velocity component can be represented as

$$v_{II} = \pm \frac{|v|}{\sqrt{1 + \varepsilon \cos \theta}} \sqrt{G + \varepsilon \cos \theta} , \qquad (2)$$

where $G = 1 - \mu B_0 / E$.

When the constant C in (1) is chosen from the condition that the particle orbit intersect the magnetic surface of radius r_1 at the point θ_1 we can change the variables as

$$x = \varepsilon \varsigma^{-\frac{2}{3}} (1+G)^{-\frac{1}{3}}, \tag{3}$$

$$\theta = \theta, \tag{4}$$

$$\psi_1 = G \varsigma^{-\frac{2}{3}} (1+G)^{-\frac{4}{3}}, \tag{5}$$

where x is the dimensionless radius.

We neglect the term on the order of ε^2 and use (3)-(5) to obtain from (1)

$$((x^2 - x_1^2)^2 + 2\sigma\Psi(x^2 - x_1^2) + x_1\cos\theta_1 - x\cos\theta = 0, \tag{6}$$

where $\sigma = \pm 1$ specifies the direction of the longitudinal velocity component with respect to the magnetic field lines and $\Psi = \sqrt{\psi_1 + x_1\cos\theta_1}$ is the normalised pitch angle at the point (x_1, θ_1).

Unlike (1), equation (6) depends only on two parameters : the position of the initial point (x_1, θ_1) and the quantity $\sigma\Psi$; consequently, the orbit topology becomes far simpler to analyse over the entire plasma column.

The ***fours-order*** equation (6) with respect to x is written in a fixed coordinate system associated with the magnetic axis. In the coordinates that are associated with a particular trajectory and have the origin at an arbitrary point on this trajectory, equation (6) passes over to a ***cubic*** equation with respect to the radial variable.

Since the drift trajectories are symmetric with respect to the equatorial plane, it is convenient to analyse the orbit topology by assuming $\theta_1 = 0$ or $\theta_1 = \pi$. Under this assumption, the quantity x_1, which will be denoted by λ, is merely the parameter of the problem such that $-\infty \leq \lambda \leq \infty$. We change the variables,

$$x^2 - \lambda^2 = z^2 + 2\lambda\cos\alpha,$$
$$x\cos\theta = z\cos\alpha + \lambda \tag{7}$$

to transform (6) to the ***cubic*** equation

$$z^3 + 4\lambda\cos\alpha z^2 + 2(2\lambda^2\cos^2\alpha + \sigma\Psi_\lambda)z + (\sigma 4\Psi_\lambda\lambda - 1)\cos\alpha = 0, \tag{8}$$

where $\Psi_\lambda = \sqrt{\psi_1 + \lambda}$.

The standard change of variable,

$$y = z + \frac{4}{3}\lambda\cos\alpha \tag{9}$$

puts equation (8) in the canonical form

$$y^3 + 3py + 2q = 0, \tag{10}$$

where

$$p = \sigma\frac{2}{3}\psi_\lambda - \frac{4}{9}\lambda^2\cos^2\alpha, \tag{11}$$

$$q = -\frac{8}{27}\lambda^3\cos^3\alpha + \sigma\frac{2}{3}\Psi_\lambda\lambda\cos\alpha - \frac{1}{2}\cos\alpha. \tag{12}$$

Hence, the topology of drift orbits described by the forth-order equation (1), which depends on four parameters, can be analysed by means of cubic equation involving two parameters.

The analytic solutions to the cubic equation (10) depend on the sign of the discriminant $D = p^3 + q^2$ and may be found elsewhere. The real positive solutions in which we are interested here are as follows:

(i) For $p = 0$ we have the trivial solution

$$y = (-2q)^{1/3}. \qquad (13)$$

(ii) For $p < 0$ and $D \leq 0$, we have

$$y_1 = -2r \cos \frac{\varphi}{3}, \qquad (14)$$

$$y_2 = +2r \cos \frac{\pi - \varphi}{3}, \qquad (15)$$

$$y_3 = +2r \cos \frac{\pi + \varphi}{3}, \qquad (16)$$

where $\cos \varphi = qr^{-3} = \beta$ and $r = \pm \sqrt{|p|}$ (the sign of r should be the same as the sign of q).

(iii) For $p < 0$ and $D < 0$, we have

$$y = -2r \cosh \frac{\varphi}{3}, \qquad \cosh \varphi = \beta. \qquad (17)$$

(iiii) For $p > 0$, we have

$$y = -2r \sinh \frac{\varphi}{3}, \qquad \sinh \varphi = \beta. \qquad (18)$$

Although this solutions has a simple form in the coordinate system that was chosen in a special way, the topology of drift orbits in real space is fairly complicated.

Now we will present some examples of the orbits which are not widely known.

Let us name the particles whose longitudinal velocity component changes sign or vanishes at least at one point along the trajectory as *trapped* particles, even if their orbits encircle the magnetic axis. The particles whose longitudinal velocity does not change sign along the trajectories will be referred to as *circulating* particles, even if their orbits do not encircle the magnetic axis.
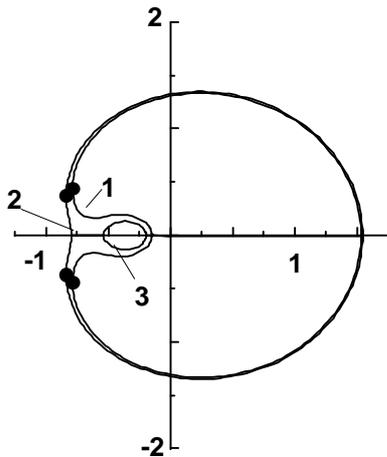

Fig.1 Drift orbits of particles with $\sigma = -1$, $\Psi_\lambda = 0.8$, $\lambda = -0.15$ (1), $\lambda = -0.2$ (2 and 3)

Some results of our analysis are represented in Figs.1-3. Full circles in this Figs. show the points, where $v_{II} = 0$.

In Fig.1 one can see the orbits with $\sigma = -1$ and $\Psi_\lambda = 0.8$. For $\lambda = -0.15$ we have the orbit of trapped particles which encircle the magnetic axe and have two turning points in each half-plane (curve 1). If we shift value of λ up to $\lambda = -0.2$ the orbit split on two - on the orbit of trapped particles which has no turning points (curve 2) and on the orbit of circulating particles which do not encircle the magnetic axe.

In Fig.2 one can see two orbits of trapped particles with $\lambda = 1.5$ and $\Psi_\lambda = 1.5$. Orbit 1 corresponds to $\sigma = -1$ and orbit 2 corresponds to $\sigma = +1$. The form of orbit 1 is the same as in the neoclassical theory – banana

trajectory, the form of orbit 2 is very difficult to refer as banana.

Figure 3 illustrates the trajectories traced by a particle that start from the magnetic surface with the radius $\lambda = 0.5$ at different values of Ψ_λ. For $\Psi_\lambda < 0.5$ the trajectories with $\sigma = \pm 1$ both pass to the right of starting point (orbit 1 for $\sigma = -1$ and $\Psi_\lambda = 0.25$ and orbit 2 for $\sigma = +1$ and $\Psi_\lambda = 0.25$). For larger values of Ψ_λ, the particle orbits lie on both sides of starting point (orbit 3 for $\sigma = -1$ and $\Psi_\lambda = 1.3$ and orbit 4 for $\sigma = +1$ and $\Psi_\lambda = 1.3$). The circle 5 represents the magnetic surface with radius $\lambda = 0.5$. From this Fig. one can see that the type of orbit may depend on the sign of σ. For the same value of normalised pitch-angle Ψ_λ one particle may be trapped (orbits 1 and 3) and another may be circulating (orbits 2 and 4).

So, in this paper we have obtained the cubic equation which depends only on two parameters. This equation makes possible to describe particle orbits in axisymmetric tokamak over the entire plasma column.
It was be shown, that we have the trapped particles whose orbits encircle the magnetic axis and have one or two turning points in each half-plane or none of them. Also, there exist circulating particles whose trajectories have turning points and do not encircle the magnetic axis.

Unlike in the neoclassical theory , we have considered a more realistic situation in which the main changes in the orbit
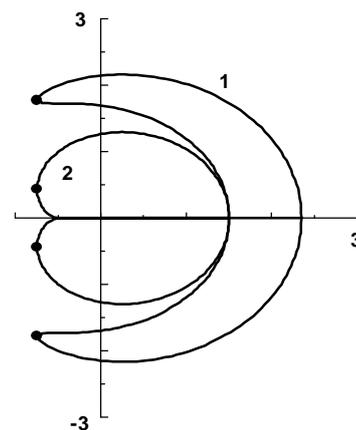are associated both with the value of the initial pitch angle of a particle and with the **sign of σ** .

Fig.2 Drift orbits of particles with $\lambda = 1.5$, $\Psi_\lambda = 1.5$, $\sigma = - 1(1)$, $\sigma = + 1 (2)$
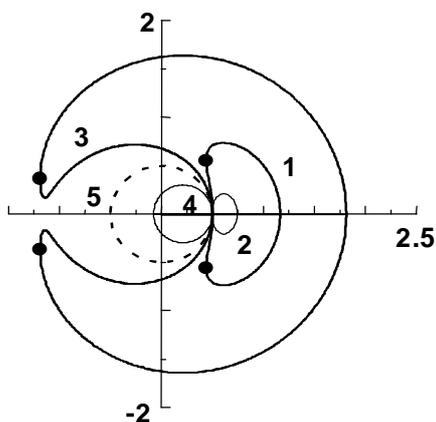

Fig.3. Drift orbits of particles with $\lambda = 0.5$
1 - $\Psi_\lambda = 0.25$, $\sigma = -1$, 2 - $\Psi_\lambda = 0.25$, $\sigma = +1$, 3 - $\Psi_\lambda = 1.3, \sigma = -1$, 4 - $\Psi_\lambda = 1.3$, $\sigma = +1$, 5 - surface with $\lambda = 0.5$.

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