Variational Principle for Electron Magnetohydrodynamics [1]

V. I. Ilgisonis and V. P. Lakhin

Russian Research Centre "Kurchatov Institute", Moscow 123182, Russia

1. Electron magnetohydrodynamics (EMHD) provides a fluid description of the plasma behaviour on scales below the ion inertial length c/ω_{pi}. As the time scales are concerned, the EMHD limit corresponds to rather fast processes when ions are practically fixed forming a static charge neutralizing background for the electrons [1, 2]. In this case the plasma dynamics is governed by the electric currents strictly related to the electron flows and their self-consistent magnetic field. EMHD model is often applied to describe the phenomena, for which the Hall effect is significant, e.g., z-pinch dynamics, magnetic field generation in laser-produced coronas, plasma erosion switches, etc. EMHD equations are also suitable to simulate current-vortex structures in plasma discharges, see, e.g., Ref. [3], where the existence and stability of some 2-D vorticies were analyzed.

However, it seems that EMHD might have also wider applications, in particular, for fusion plasmas. Suppose there is a conventional plasma equilibrium, which is stable with respect to ideal MHD perturbations. If not, although the real plasma is doubtless dissipative one, fast MHD processes are able to change drastically the magnetic configuration on the Alfvén time scales, that is inappropriate for fusion purposes. The typical EMHD time scales are much shorter, therefore, even if the plasma configuration was initially MHD- stable, the electron instability can also perturb the plasma current and magnetic field, creating new conditions for the development of MHD instability. E.g., a disruption in a tokamak is usually preceeded by the burst of magnetic plasma activity, so-called "precursor", whose nature is still unknown (see, e.g., Ref. [4]). One cannot exclude that just fast EMHD processes are responsible for the change of magnetic field configuration which earlier was MHD stable. To study such a possibility, one has to investigate the electron stability conditions for the configuration, which is in an equilibrium in the frames of both conventional and electron MHD models.

2. EMHD is known to be formed by the following set of equations:

$$\partial_t \mathbf{B}_* = \boldsymbol{\nabla} \times [\mathbf{v} \times \mathbf{B}_*] + a^{-1}\boldsymbol{\nabla}T \times \boldsymbol{\nabla}\eta, \tag{1}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = a\rho\mathbf{v}, \tag{2}$$

$$\mathbf{B}_* = \mathbf{B} + a^{-1}\boldsymbol{\nabla} \times \mathbf{v}, \tag{3}$$

$$\partial_t \eta + \mathbf{v} \cdot \boldsymbol{\nabla}\eta = 0. \tag{4}$$

Here the first equation is obtained by applying *curl* to the equation of the motion of the electron fluid, Eq.(2) is Maxwell equation, Eq.(4) is the adiabatic equation. On the whole the symbols are standard, $a = e/m$ is the electron charge-to-mass ratio, η is the specific entropy and T is the temperature of electrons. Plasma is considered to be quasineutral, and since the ions are assumed to be fixed, then the electron density does not change in time, $\partial_t \rho = 0$. It is equal to the ion background density, so that $\rho = \rho(\mathbf{x})$ (the effects of

a density perturbation are discussed in Ref. [5]). The set of Eqs.(1)–(4) is closed if the temperature is defined as the function of density and entropy, $T = T(\rho, \eta)$. Below we shall restrict ourselves to the important case of a barotropic electron fluid, $T = T(\eta)$, which cancels the last term in Eq.(1) and hence allows us to omit the adiabatic equation (4). Excluding $\mathbf{v}$, we may keep the only equation

$$\partial_t \mathbf{B}_* = \boldsymbol{\nabla} \times \left[\frac{(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}_*}{a^2 \rho} \right] , \tag{5}$$

considering the relation

$$\mathbf{B}_* = \mathbf{B} + \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} \times \mathbf{B}}{a^2 \rho} \right) \tag{6}$$

as the definition of $\mathbf{B}_*$. It is also applicable for a case of "cold" electrons with negligible electron pressure, and for an isentropic case ($\nabla \eta = 0$) as well.

Since EMHD equation (5) takes its origin from two-fluid hydrodynamics, it can be derived from the Lagrangian

$$L = \int d^3\mathbf{x} \left\{ \frac{\rho \mathbf{v}^2}{2} + a\rho \mathbf{v} \cdot \mathbf{A} - \frac{\mathbf{B}^2}{2} \right\} , \tag{7}$$

by the variation of L over two vector potentials $\mathbf{A}$ and $\boldsymbol{\chi}$, and by use the definition (6). Here $\boldsymbol{\chi}$ determines the electron fluid displacement $\boldsymbol{\xi}$ to avoid the perturbation of the density ρ, and $\mathbf{A}$ determines the magnetic field,

$$\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B} , \qquad \boldsymbol{\nabla} \times \boldsymbol{\chi} = a\rho \boldsymbol{\xi} . \tag{8}$$

In Ref. [6] all the first order Lie-Bäcklund symmetries of EMHD equations were found as well as the corresponding conservation laws. Eq. (5) appears to be completely equivalent to 3 scalar conservation laws resulting from the relabeling symmetry. It is rather obvious since $\mathbf{B}_*$, which satisfies Eq. (5), is a frozen-in vector carried by the electron fluid. Such a dynamics can be integrated in terms of Lagrangean coordinates

$$\alpha^i(t, \mathbf{x}) : \quad \partial_t \alpha^i + \mathbf{v} \cdot \boldsymbol{\nabla} \alpha^i = 0, \quad i = 1, \ 2, \ 3 ,$$

associated with the electron fluid, namely,

$$\mathbf{A}_* = \sum_{j=1}^{3} A_{*j}(\{\alpha^i\}) \boldsymbol{\nabla} \alpha^j ; \quad \mathbf{B}_* = \boldsymbol{\nabla} \times \mathbf{A}_* . \tag{9}$$

It is important to emphasize that Eq. (9) presents a general solution to Eq. (5), therefore, in contrast with Ref. [3], there is no necessity to take care of other conservation laws of Eq. (5) (of the so-called Casimirs, the infinite number of which can be generated).

3. To investigate stability of an arbitrary EMHD equilibrium, we use a variational approach. Consider Hamiltonian

$$H = \int d^3\mathbf{x} \left\{ \frac{\rho \mathbf{v}^2}{2} + \frac{\mathbf{B}^2}{2} \right\} = \int d^3\mathbf{x} \left\{ \frac{\rho a^2}{2} (\mathbf{A}_* - \mathbf{A})^2 + \frac{(\boldsymbol{\nabla} \times \mathbf{A})^2}{2} \right\} \tag{10}$$

subject to the independent variations over $\mathbf{A}$ and $\boldsymbol{\chi}$:

$$\delta \mathbf{B}_* = \boldsymbol{\nabla} \times [\boldsymbol{\xi} \times \mathbf{B}] = \boldsymbol{\nabla} \times \left[\frac{\boldsymbol{\nabla} \times \boldsymbol{\chi}}{a\rho} \times \mathbf{B}_* \right] , \tag{11}$$

$$\delta \mathbf{B} = \boldsymbol{\nabla} \times \delta \mathbf{A} . \tag{12}$$

Being the constant of the motion, H may be considered as Lyapunov functional. First variation of Eq. (10) gives the general equilibrium:

$$\delta H = \int d^3\mathbf{x}\{\delta\mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B} - a\rho\mathbf{v}) - \boldsymbol{\chi} \cdot \boldsymbol{\nabla} \times [\mathbf{v} \times \mathbf{B}_*]\} \Rightarrow \tag{13}$$

(here $\mathbf{v}$ simply denotes $a(\mathbf{A}_* - \mathbf{A})$)

$$\Rightarrow \delta H = 0 \quad \Leftrightarrow \quad (\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}_* = \rho\boldsymbol{\nabla}\Phi \;, \tag{14}$$

Φ - arbitrary function. Taking the second variation and minimizing over $\delta\mathbf{A}$, we find

$$\delta^2 H = \int d^3\mathbf{x}\, \delta\mathbf{B}_* \cdot \{\delta\mathbf{B} + (\boldsymbol{\nabla} \times \mathbf{B}) \times \boldsymbol{\xi}\} \;. \tag{15}$$

Here $\delta\mathbf{B}_*$ is given by Eq. (11), and $\delta\mathbf{B}$:

$$\delta\mathbf{B} + \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} \times \delta\mathbf{B}}{a^2\rho}\right) = \delta\mathbf{B}_* \;. \tag{16}$$

It is important to emphasize that $\delta^2 H$ takes its minimal value for $\delta\mathbf{A}$ (and, therefore, for $\delta\mathbf{B}$) satisfying Eq. (16), which is exactly the definition (6) in variations. However, this equation cannot be solved analytically in a general form.

4. The only parameter involved into Eq. (16) is the ratio d_e/d, where $d_e = \frac{1}{\sqrt{a^2\rho}} = \frac{c}{|\omega_{pe}|}$ is the collisionless skin depth, and d is the typical scale length of the perturbed magnetic field. For a fusion plasma with the concentration $\rho/m_e \approx 10^{14}\text{cm}^{-3}$, $d_e^2 \approx 2.8\cdot10^{-3}\text{cm}^{-2}$, and the ratio $(d_e/d)^2$ may be much smaller than 1. The limit $(d_e/d)^2 \to 0$ corresponds to the negligible contribution of electron inertia to dynamic Eq. (5), which becomes to be the frozen-in equation for the magnetic field, and $\delta\mathbf{B}_* \to \delta\mathbf{B}$. In this limit the stability criterion

$$\delta^2 H_0 \approx \delta^2 H\Big|_{d_e/d\to 0} \approx \int d^3\mathbf{x}\{(\boldsymbol{\nabla} \times [\boldsymbol{\xi} \times \mathbf{B}])^2 + \boldsymbol{\nabla} \times [\boldsymbol{\xi} \times \mathbf{B}] \cdot [(\boldsymbol{\nabla} \times \mathbf{B}) \times \boldsymbol{\xi}]\} > 0 \tag{17}$$

looks exactly as the MHD energy principle [7] taken for a negligible plasma pressure or compressibility. For some 2D vortex configurations it was analysed in Ref. [3]. For a tokamak equilibrium it corresponds to the known kink mode stability condition (because the most dangerous kink mode is incompressible one) and, therefore, gives nothing new for a case when MHD stability is already guaranteed.

Suppose $(d_e/d)^2$ is small but finite quantity allowing for the next order solution of Eq. (16), namely,

$$\delta\mathbf{B} \approx \delta\mathbf{B}_* - \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} \times \delta\mathbf{B}_*}{a^2\rho}\right) \;. \tag{18}$$

It results in the following stability condition

$$0 < \delta^2 H_1 \approx \int d^3\mathbf{x} \left\{\delta\mathbf{B}_*{}^2 + \delta\mathbf{B}_* \cdot [(\boldsymbol{\nabla} \times \mathbf{B}) \times \boldsymbol{\xi}] - \frac{(\boldsymbol{\nabla} \times \delta\mathbf{B}_*)^2}{a^2\rho}\right\} \;, \tag{19}$$

where $\delta\mathbf{B}_*$ via $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$ is given by Eq. (11) as earlier. The last term in Eq. (19) makes an explicit difference of our criterion from the MHD one. It corresponds to the contribution

of the perturbed electron motion (current), which always appears to be destabilizing. As we were looking for a structure of the most dangerous perturbations, we have kept only the volume part of the integrals in Eqs. (17), (19).

5. The peculiarity of criterion Eq. (19) is that the last destabilizing term, first, contains the high order derivative of the perturbation, and second, is weighted with the factor $(d_e/d)^2$ assumed to be small. Hence the simplest conclusion that an instability can always develop when the wavelength number k will be sufficiently large, $|kd_e|^2 \sim 1$, contradicts to our assumption $k \sim 1/d \leq 1/d_e$. Indeed, the exact condition $\delta^2 H > 0$, where $\delta^2 H$ is given by Eq. (15), generally results in the sign-alternative series

$$0 \; < \; \delta^2 H_1 + \int d^3\mathbf{x} \left\{ \left(\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} \times \delta\mathbf{B}_*}{a^2\rho} \right) \right)^2 - \frac{1}{a^2\rho} \left(\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} \times \delta\mathbf{B}_*}{a^2\rho} \right) \right)^2 + \dots \right\} \; ,$$

which cannot be cut at any term in the case of $(d_e/d)^2 > 1$.

However, our criterion Eq. (19) may indicate the possible electron instability of the plasma equilibrium, which is near a threshold with respect to any incompressible MHD-perturbations. That instability would be able to diminish the perturbation wavelength and to tend to a current filamentation. Nevertheless, a final conclusion of how that instability develops (if it does), can be made by numerical simulations. Note that in the case of $\rho = $ const, the unknown quantity $\delta\mathbf{B}$ can be restored from Eq. (18) via the Green function:

$$\delta\mathbf{B} = \int d^3\mathbf{x}_1 \left(\mathbf{B}_*(\boldsymbol{\xi} \cdot \boldsymbol{\nabla})G - \boldsymbol{\xi}(\mathbf{B} \cdot \boldsymbol{\nabla})G \right) \; , \tag{20}$$

where

$$G(\mathbf{x}, \mathbf{x}_1) = \frac{\exp(-|\mathbf{x} - \mathbf{x}_1|d_e^{-1})}{4\pi|\mathbf{x} - \mathbf{x}_1|d_e^2} \; , \tag{21}$$

and all the vectors in the integrand are defined at the $\mathbf{x}_1$-space.

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