

Multi-mode Parametric Excitation of Bernstein Waves under Electron Cyclotron Heating

A.A. Ivanov¹, K.S. Serebrennikov¹, V.Yu. Fedotov¹, M. Bacal²,
J.-M. Buzzi².

¹Russian Research Center "Kurchatov Institute", Moscow, Russia

²Laboratoire de Physique des Milieux Ionises, UMR 7648 du C.N.R.S., Ecole Polytechnique,
91128 Palaiseau Cedex, France

If a homogeneous plasma undergoes an influence of a potential electric field $\vec{E} = \vec{E}_0 \sin \omega_0 t$ perpendicular to an external magnetic field \vec{B} , then an instability is excited in the plasma. One can represent the time development of the instability as Floque series

$$\varphi = \exp(i\omega t) \sum a_n \exp(i\omega_0 t)$$

here φ is a potential of the excited potential waves [1]

Substituting the series in Vlasov's equation we obtain an infinite system of algebraic equations in a_n . The solvability condition for the system gives the following equation for ω

$$D(\omega, \vec{k}) = \begin{vmatrix} I & D^e \\ D^i & I \end{vmatrix} = 0$$

here $D(\omega, \vec{k})$ is a block matrix, I is the unit matrix, D^e and D^i are blocks with the following elements: $D_{nm}^e = R_e^{(n)} J_{n-m}(\mu)$, $D_{nm}^i = R_i^{(n)} J_{m-n}(\mu)$; $\mu = |\vec{k}\vec{b}|$, \vec{k} is the wave vector of the excited wave, \vec{b} is the amplitude of electron oscillations, Ω is the electron gyro frequency,

$$R_\alpha^{(n)} = \frac{\delta \varepsilon_\alpha(\omega + n\omega_0, \vec{k})}{1 + \delta \varepsilon_\alpha(\omega + n\omega_0, \vec{k})}, \quad \alpha = e, i.$$

If $\vec{k} \perp \vec{B}$ and one put ions to be nonmagnetized then

$$\delta \varepsilon_i = -\frac{\omega_{pi}^2}{\omega^2} \quad \text{and} \quad \delta \varepsilon_e(\omega, k) = -2 \sum_{n=1}^{\infty} \frac{n^2 \Omega^2 \Phi_n(k)}{\omega^2 - n^2 \Omega^2}$$

here $\Phi_n = I_n(k^2 \rho_e^2) \exp(-k^2 \rho_e^2) / k^2 r_{De}^2$, ω_{pi} is the ion plasma frequency, r_{De} and ρ_e are the electron Debye and gyro radii, I_n is the modified Bessel function.

Having followed [2] and [3] we shall simplify equation (1). One can easily prove that the infinite determinant $D(\omega, \vec{k})$, regarded as a function of complex variable ω , appears to be even and periodic with a period equal to ω_0 . Elements of the matrix $D(\omega, \vec{k})$ have no singularities, but for poles. The poles are roots of the equation $1 + \delta \varepsilon_\alpha(\omega, \vec{k}) = 0$. Then according to Mittag-Leffler's theorem $D(\omega, \vec{k})$ can be represented as follows

$$D(\omega, k) = N(\omega) + \sum_\alpha \sum_n \frac{K_{\alpha n}}{\sin^2(\pi\omega/\omega_0) - \sin^2(\pi\omega_{\alpha n}/\omega_0)},$$

where $N(\omega)$ is an entire periodic function, $K_{\alpha n} = \frac{\pi D_{\alpha n} \sin(2\pi\omega_{\alpha n}/\omega_0)}{\omega_0 \partial \delta \varepsilon_\alpha(\omega, k) / \partial \omega \Big|_{\omega=\omega_{\alpha n}}}$, $\omega_{\alpha n}$ are the

poles of $D(\omega, \vec{k})$, and the coefficients D_{ω_n} are obtained by substituting ω_{ω_n} in $D(\omega, \vec{k})$ with subsequent regularization of the string containing the pole ω_{ω_n} . Easy test shows that $N(i\infty) = 1$ and by Liouville's theorem $N(\omega) = 1$

Therefore analytical properties of the determinant $D(\omega, \vec{k})$ permit us to express it in terms of simple periodic functions of ω and coefficients D_{ω_n} independent on ω . Taking into consideration that $\omega_{pi} / \omega_0 \ll 1$ we obtain up to the order of $(\omega_{pi} / \omega_0)^2$

$$D(\omega, k) = 1 + \frac{\pi\omega_{pi}}{2\omega_0} \sum_{n=-\infty}^{\infty} R_e^{(n)}(\omega)(-1)^n \left(\frac{J_{n+\lambda^-}(\mu)J_{-n-\lambda^-}(\mu)}{\sin(\pi\lambda^-)} - \frac{J_{n+\lambda^+}(\mu)J_{-n-\lambda^+}(\mu)}{\sin(\pi\lambda^+)} \right) \quad (1)$$

here $\lambda^\pm = \frac{\omega \pm \omega_{pi}}{\omega_0}$.

This formula is valid for arbitrary values of ω .

1. Dispersion equation for arbitrary wave length

Denote $\omega = i\gamma$, $K = kp_e$, $\Delta = \omega_0 - \Omega$ and $R = r_{De} / \rho_e = \Omega / \omega_{pe}$, $a = \frac{1}{2} \frac{E_0}{B} \frac{c}{v_{Te}}$,

and let $R \gg 1$, $\omega / \Omega \ll 1$, and $\delta = \Delta / \Omega \ll 1$.

Then the positive roots of the equation $1 + \delta \varepsilon_e(\omega, K) = 0$ are

$$\omega_{en} = n\Omega(1 + \Phi_n(K)),$$

Now expression (1) can be simplified, and we obtain the following dispersion equation

$$1 - \frac{\omega_{pi}^2}{\gamma^2 + \omega_{pi}^2} \sum_{n=1}^{\infty} \frac{2n^2 \Omega (\Omega \Phi_n - \Delta) \Phi_n}{\gamma^2 + n^2 (\Delta - \Omega \Phi_n)^2} J_n^2(\mu) = 0 \quad (2)$$

here $\mu \approx \frac{a}{\delta} K$.

2. Dispersion equations for long and short wave lengths

If $K \gg 1$ (short wave lengths) then (2) can be reduced to

$$1 - \frac{\omega_{pi}^2}{\gamma^2 + \omega_{pi}^2} \frac{\Phi(K)}{\Phi(K) - \delta} \left(1 - \frac{\pi v}{\sinh \pi v} J_{iv}(\mu) J_{-iv}(\mu) \right) = 0 \quad (3)$$

Here $v = \gamma / (\Omega \Phi - \Delta)$.

This equation contains contribution of all of the Bernstein modes. We search positive solutions of equation (3), i.e. $\gamma > 0$.

If $K \ll 1$ (long wave lengths) then $\Phi_n / \Phi_1 \ll 1$, and we keep in (2) the terms with $n = 1$ only.

$$1 - \frac{\omega_{pi}^2}{\gamma^2 + \omega_{pi}^2} 2\Omega^2 J_1^2(\mu) \frac{(\Phi_1(K) - \delta)\Phi_1(K)}{\gamma^2 + \Omega^2 (\Phi_1(K) - \delta)^2} = 0 \quad (4)$$

3. Strong electric fields

Strong electric fields are characterized by the inequality

$$a > \Phi_1(0) = \frac{1}{2R^2} \text{ i.e. } eE_0 > mv_{Te} \frac{\omega_{pe}^2}{\Omega}$$

The range of values of δ is split up into two physically different domains. The first is

$$0.3\Phi_1(0) = \delta_1 < \delta < \delta_0 = \Phi_1(0) \text{ i.e. } 0.15 \frac{\omega_{pe}^2}{\Omega} < \Delta < \frac{1}{2} \frac{\omega_{pe}^2}{\Omega} \quad (5)$$

In this range long waves ($K = kp_e < 1$) are excited.

The second domain is

$$\delta < \delta_1 \text{ i.e. } \Delta < 0.15 \frac{\omega_{pe}^2}{\Omega} \quad (6)$$

In this range short waves ($K = k\rho_e > 1$) are excited.

Given δ external field excites waves with various values of wave vector. We shall search wave vectors with maximal linear growth rate γ , i.e. we must find from (3) and (4) the linear growth rate and solve the equation $\partial\gamma/\partial K = 0$.

3.1 Long waves

For the case of long waves (equation (4) and inequality (5)) we have the following expression for the maximal growth rate γ_m

$$\gamma_m^2 = -\frac{1}{2}\Omega^2(\Phi_1(0)-\delta)^2 - \frac{1}{2}\omega_{pi}^2 + \frac{1}{2}\sqrt{\left(\Omega^2(\Phi_1(0)-\delta)^2 + \omega_{pi}^2\right)^2 + 4\Omega^2\omega_{pi}^2(\Phi_1(0)-\delta)(\delta-0.3\Phi_1(0))}$$

The wave vector corresponding to the maximal growth rate is

$$K_m = 1.84 \frac{\delta}{a} \ll 1 \quad \mu_m = 1.84$$

γ_m as a function of δ has a maximum. So the growth rate as function of K and δ has the absolute maximum Γ .

$$\Gamma \approx 0.55\omega_{pi} \left(\frac{\omega_{pe}^2}{\Omega\omega_{Ti}} \right)^{1/3}$$

This value is achieved at the following values of δ and K

$$\delta = \tilde{\delta} = \frac{1}{2} \frac{\omega_{pe}^2}{\Omega^2} - 0.55 \frac{\omega_{pi}}{\Omega} \left(\frac{\omega_{pe}^2}{\omega_{Ti}\Omega} \right)^{1/3} \quad \text{and} \quad K = \tilde{K} = 1.84 \frac{\tilde{\delta}}{a}$$

3.2 Short waves

For the case of short waves (equation (3) and inequality (6)) the maximal growth rate and corresponding wave vector have the form

$$\gamma_m = \frac{2}{3} \omega_{pi} \left(\frac{a\Omega}{\omega_{pi}} \right)^{1/3} \left(\frac{\omega_{pe}^2}{\sqrt{2\pi}\Omega^2} \frac{1}{\delta} \right)^{1/9} \quad (7)$$

and

$$K_m = \left(\frac{\omega_{pe}^2}{\sqrt{2\pi}\Omega^2} \frac{1}{\delta} \right)^{1/3} - \frac{1}{3} \left(0.7 \frac{\omega_{pi}}{a\Omega} \right)^{2/3} \left(\frac{\omega_{pe}^2}{\sqrt{2\pi}\Omega^2} \frac{1}{\delta} \right)^{1/9}$$

Substitution of the maximal growth rate (7) in equation (2) shows that approximately $aK_m/\delta (\gg 1)$ summands make equal contribution to the dispersion equation and, therefore, a lot of Bernstein modes are excited.

4. Weak electric fields

Now consider weak electric field

$$a \ll \Phi_1(0) = \frac{1}{2R^2} \text{ i.e. } eE_0 \ll mv_{Te} \frac{\omega_{pe}^2}{\Omega}$$

In this case the range of values of δ is split up into three domains

$$a^{3/4}(\Phi_1(0))^{1/4} = \delta_1 < \delta < \delta_0 < \Phi_1(0) \text{ i.e. } a^{3/4} \left(\frac{\omega_{pe}\Omega}{\sqrt{2}} \right)^{1/2} < \Delta < \frac{1}{2} \frac{\omega_{pe}^2}{\Omega} \quad (8)$$

$$\frac{a^3\Omega^3}{\omega_{pi}^3} \Phi_1(0) = \delta_2 < \delta < \delta_1 \text{ i.e. } \frac{a^3\Omega^2\omega_{pe}^2}{2\omega_{pi}^3} < \Delta < a^{3/4} \left(\frac{\omega_{pe}\Omega}{\sqrt{2}} \right)^{1/2} \quad (9)$$

$$\delta < \delta_2 \text{ i.e.} \quad \Delta < \frac{a^3 \Omega^2 \omega_{Te}^2}{2\omega_{pi}^3} \quad (10)$$

The growth rate achieves the maximal value γ_m at the following value of K

$$K_m = \left(\frac{\omega_{pe}^2}{\sqrt{2\pi}\Omega^2} \frac{1}{\delta} \right)^{1/3} - \frac{1}{3} \left(0.7 \frac{\omega_{pi}}{a\Omega} \right)^{2/3} \left(\frac{\omega_{pe}^2}{\sqrt{2\pi}\Omega^2} \frac{1}{\delta} \right)^{1/9} \quad \text{for region (8) and (10)}$$

$$K_m = \left(\frac{a\omega_{pe}^2}{\omega_{pi}\Omega} \frac{1}{\sqrt{2\pi}\delta} \right)^{1/2} \quad \text{for region (9)}$$

Corresponding values of γ_m are as follows

$$\gamma_m = \frac{a^2 K_m^2 \Omega}{4\delta} \quad \text{for region (8)}$$

$$\gamma_m = a K_m \Omega \quad \text{for region (9)}$$

$$\gamma_m = \frac{2}{3} \omega_{pi} \left(\frac{a\Omega}{\omega_{pi}} \right)^{1/3} \left(\frac{\omega_{pe}^2}{\sqrt{2\pi}\Omega^2 \delta} \right)^{1/9} \quad \text{for region (10)}$$

Conclusion

The above account shows that the linear growth rate of parametric excitation of the Bernstein modes may sufficiently exceed ω_{pi} even when the external pumping field is weak, The growth rate and the wave vector of the excited waves increases when $\delta (= \Delta / \Omega)$ decreases.

It should be observed that the growth rate of parametrically excited waves propagating nearly parallel to external magnetic field is less than ω_{pi} [4,5]. Therefore the Bernstein modes make a dominant contribution to development of the parametric instability in the vicinity of the gyro frequency.

The most interesting case is one of strong pumping fields and small δ . In this case short waves are excited and growth rate is more than ω_{pi} . But unlike weak fields a lot of Bernstein modes are excited simultaneously.

In conclusion notice an important feature of the mathematical formalism employed. Unlike many others authors we were not restricting ourselves to the condition $\mu < 1$. This condition corresponds to very long waves. Really, even if $k\rho_e < 1$ the maximal growth rate is achieved at $\mu \approx 2$ (see section 3.1).

References

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