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Three-dimensional toroidal magnetic fields with islands: analytical examples and numerical model with scalar functions

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1. Analytical examples of 3D toroidal magnetic fields. Analytic 3D plasma equilibrium configurations with the current density $\mathbf{j} = \lambda\mathbf{B}$, $\lambda = constant$ can be obtained as a superposition of several 2D magnetic fields with one coordinate of symmetry (planar, axial or helical). Each of them is given by the following "mixed" representation of the magnetic field:

$$2\pi\mathbf{B} = \nabla\Psi \times \mathbf{b} + F\mathbf{b}, \quad F = F(\Psi) \tag{1}$$

where for specific force free configurations $F = \lambda\Psi$, the base vector $\mathbf{b}$ and the exact solution Ψ of the corresponding Grad-Shafranov equation are known.

In particular, for axisymmetric force-free configurations, $\mathbf{B} = \mathbf{B}_{ax}(r,z), \mathbf{b} = \mathbf{b}_{ax} = \nabla\phi$, the following partial solution can be obtained for the poloidal flux function $\Psi = \Psi_{ax}$:

$$\Psi_{ax}(r,z) = rJ_1(\lambda_2 r)\cos(\lambda_1 z), \quad \lambda_1^2 + \lambda_2^2 = \lambda^2. \tag{2}$$

The up-down symmetric system of isolines of the function (2) has a set of "o-points" (magnetic axes) in equatorial planes $\lambda_1 z = 0, \pm\pi, \pm 2\pi, \ldots$ with r satisfying $J_0(\lambda_2 r) = 0$. The elongation of the magnetic surfaces near the axes is $E = |\lambda_2/\lambda_1|$. An example of the axisymmetric configurations generated by (2) for the values $\lambda = 0.95, \lambda_1 = \lambda_2$ is presented in Fig. 1. For this equilibrium the safety factor q decreases from $q_0 = 0.58$ at the magnetic axis with a resonant surface $q = 0.5$ somewhere inside.

More complex combinations of the exact solutions allow to model more sophisticated configurations including, for instance, doublets. Fig. 2 demonstrates such an example corresponding to the following Ψ_{ax} function

$$\Psi_{ax}(r,z) = rJ_1(\lambda_2 r)\cos(\lambda_1 z) - 0.6rJ_1(\lambda_{2s} r)\sin(\lambda_{1s} z) \tag{3}$$

with the same $\lambda = 0.95$, $\lambda_1 = \lambda_2$ and $\lambda_{2s} = 1.2\lambda_2$, $\lambda_1^2 + \lambda_2^2 = \lambda_{1s}^2 + \lambda_{2s}^2 = \lambda^2$.

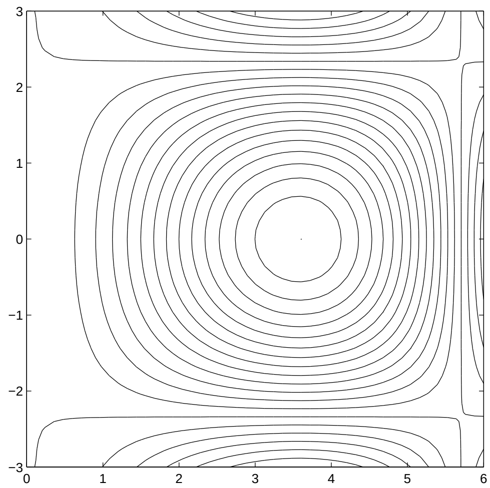


Fig. 1 Magnetic surfaces for an axisymmetric analytic equilibrium from Eq. (2).

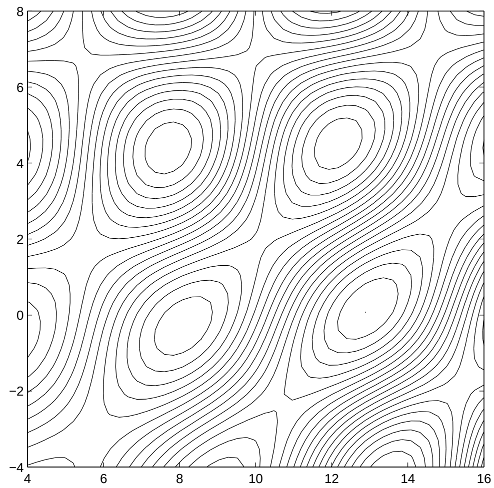


Fig. 2 Magnetic surfaces for an axisymmetric analytic equilibrium with a "figure eight" separatrix from Eq. (3).

Helically symmetric magnetic fields $\mathbf{B}_{hel}$ with the current density $\mathbf{j} = \lambda \mathbf{B}$ are described by the representation (1) with $\mathbf{b} = \mathbf{b}_{hel} = (-n/k\nabla z + r^2\nabla\phi)/(n^2/k^2 + r^2), \quad k = n_z/R$, where $2\pi R$ is a helicity period length, n and n_z are helicity numbers in ϕ and z directions respectively. The solution of the corresponding Grad-Shafranov type equation $\Psi = \Psi_{hel}$ is a function of r and $\zeta = n\phi + kz$. One of the partial solutions is the following function

$$\Psi_{hel}(r,\zeta) = \left(\frac{1}{\sqrt{|k^2 - \lambda^2|}} r I_n'(y) + \frac{n\lambda}{k(k^2 - \lambda^2)} I_n(y) \right) \cos\zeta, \quad y = \sqrt{|k^2 - \lambda^2|}r \qquad (4)$$

when $k^2 > \lambda^2$. For the case of opposite inequality, the modified Bessel function I_n in (4) should be replaced by J_n. The function (4) was used for testing the PIES code [2].

Due to the linear character of the problem considered, any linear combination of the magnetic fields $\mathbf{B}_{ax}, \mathbf{B}_{hel}$ with the same value of λ is also an equilibrium field. A lot of three-dimensional configurations can be generated in this way. By means of magnetic field line tracing one can easily examine 3D effects on the topology of magnetic surfaces.

For example, adding a small amount of $\mathbf{B}_{hel}$ to $\mathbf{B}_{ax}$ leads to three-dimensional configurations with closed surfaces: $\mathbf{B}_{3Dtor} = \mathbf{B}_{ax} + w_{hel}\mathbf{B}_{hel}$.

Fig. 3 demonstrates the results of such a superposition of the magnetic field shown in Fig. 1 and a $\mathbf{B}_{hel}$ corresponding to the flux function (4) with a weight coefficient $w_{hel} = 0.008$ and $R = 1, n = -1, n_z = 1$. The cross-sections at several toroidal angle values show the 1/2 island generated on the resonant surface $q = 0.5$. No evidence of field line stochastization was found in the region considered.

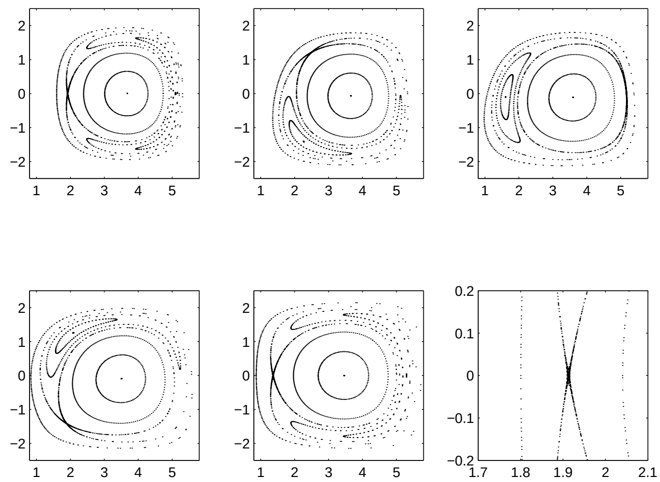
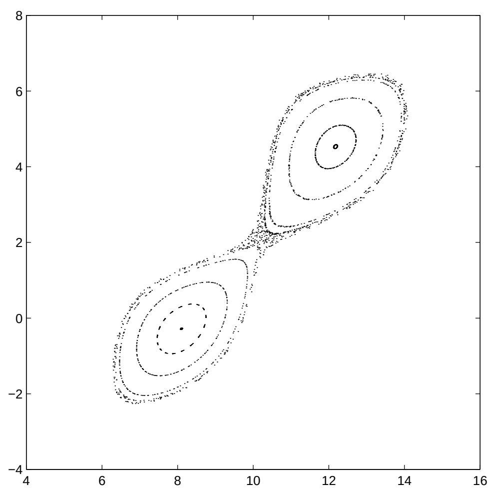


Fig. 3 Poincaré plot of the 3D toroidal magnetic field generated by a helical perturbation of the axisymmetric configuration in Fig. 1. The toroidal cross sections $\phi = 0, \pi/4, \pi/2, 3\pi/4, \pi$ and a zoom of the cross section $\phi = 0$ near the x-point are shown.



The 3D effect of stochastization of a "figure-eight" axisymmetric separatrix is presented in Fig. 4. This is a result of adding $\mathbf{B}_{hel}$ with $w_{hel} = 0.00005, n/n_z = -1/1$ to the "doublet" magnetic field from Fig. 2.

The configurations presented can be used as nontrivial tests for 3D equilibrium codes.

Fig. 4 Poincaré plot of the 3D toroidal magnetic field generated by a helical perturbation of the axisymmetric configuration in Fig. 2. The toroidal cross section $\phi = 0$ is shown.

2. Scalar function model for 3D MHD equilibrium. The same representation (1) of the magnetic field as in a symmetric case was proposed earlier [1] as a basis for the description of three-dimensional MHD equilibria with "good" (but not necessary nested) magnetic surfaces. The model includes the equations corresponding to basic properties of magnetic field $\mathrm{div}\,\mathbf{B} = 0, \mathbf{B}\cdot\nabla\Psi = 0$, $\mathrm{div}\,(\mathbf{B}\times\nabla\Psi) = 0$, and the following 3D Ψ-equation

$$\mathrm{div}\,(b^2\nabla\Psi) + \mathrm{rot}\,\mathbf{b}\cdot\mathbf{b}\times\nabla\Psi = F\mathbf{b}\cdot\,\mathrm{rot}\,\mathbf{b} - 2\pi\mathbf{j}\cdot\mathbf{b} \tag{5}$$

with a few possible variants for the representation of the current projection term $\mathbf{j}\cdot\mathbf{b}$ taking into account the force balance equation $\nabla p = \mathbf{j}\times\mathbf{B}$, $p = p(\Psi)$. In the general case, this term can be represented as follows

$$2\pi\mathbf{j}\cdot\mathbf{b} = 4\pi^2 p' + b^2 F(F' + \alpha), \tag{6}$$

the function α satisfies the magnetic differential equation: $\mathbf{B}\cdot\nabla\alpha = 2\pi p'\mathbf{q}\cdot\nabla(\ln B^2)$, $\mathbf{q} = \mathbf{b}/b^2$. In the particular case of force free equilibria considered, the term $\mathbf{j}\cdot\mathbf{b}$ in (5) takes the form $2\pi\mathbf{j}\cdot\mathbf{b} = \lambda b^2 F$.

3. Numerical experiments for force free equilibria. The determination of the magnetic flux function given a known magnetic field can be considered as a step in a 3D equilibrium solution. A formulation and numerical solution of this problem is a nontrivial task especially in the case with islands. Moreover in this case, it is not obvious what magnetic flux could be used for the magnetic field description. In a more general formulation, any scalar function Ψ satisfying the condition $\mathbf{B}\cdot\nabla\Psi = 0$ (label function) can be searched for. Together with an arbitrary choice of the function $F(\Psi)$ allowed, it leads to a violation of the closure condition for the base vector $\mathbf{b}$ that was assumed originally [1].

Equation (5) can naturally be considered as an equation for the label function Ψ assuming that the base vector $\mathbf{b}$ that satisfies the basic properties of the magnetic field is known (exactly or approximately). Corresponding numerical experiments were performed for the force free equilibrium magnetic fields described above.

In all computations, the base vector field $\mathbf{b}$ was assumed to be prescribed inside the computational domain. When the magnetic field $\mathbf{B}$ and some approximate label function Ψ_0 are known, then the vector $\mathbf{b}$ can be found from the following representation

$$\mathbf{b} = 2\pi\frac{\mathbf{B}\times\nabla\Psi_0 + F\mathbf{B}}{|\nabla\Psi_0|^2 + F^2}, \tag{7}$$

which is equivalent to (1) if $\mathbf{b}\cdot\nabla\Psi_0 = 0$. Different label functions Ψ_0 and the same function $F = constant = 1$ were mostly used in the experiments undertaken. The scalar label function Ψ can represent magnetic surfaces with its isosurfaces only if $|\nabla\Psi| = 0$ at the island magnetic axis (similar to the helical flux function). So the input label function with a small gradient at the position of the magnetic island was specified.

The following choices of input function were used in experiments gradually approaching the exact island structure:

– helical flux extrapolated from an axisymmetric configuration;

– helical flux taken from the VMEC approximation with nested surfaces;

– the VMEC helical flux modulated with a simple analytic 3D function to produce an island structure in Ψ_0;

– a "hand-made" label function produced with magnetic field line tracing on a finite grid.

The force-free 3D configuration with a big $m/n = 1/2$ island (Fig. 3) was mostly used for modeling.

All these 3D calculations demonstrate a more or less strong "sausage" effect: one (or several, in general) local extrema of the solution (with spherical isosurface topology instead of toroidal one) appear in place of a magnetic island.

The most adequate result was obtained in the last approach mentioned when the input label function was directly generated in the process of magnetic field line tracing (the "hand made" label). Fig. 5 presents the result of the use of this approach to provide the input label for the Ψ-equation. The solution isosurfaces look in this case rather close to the exact magnetic surfaces (Fig. 3). The result can be considered as a consistency check for the numerical technique applied and shows that the solution of the 3D Ψ-equation can in principle reproduce the magnetic surface label sufficiently close to the exact one. Moreover the numerical procedure tolerates input label functions that may not necessary be very smooth.

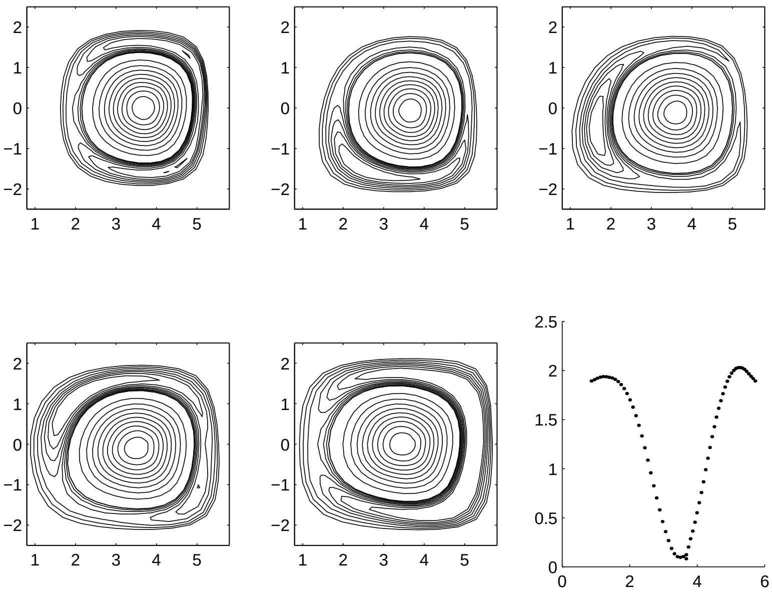


Fig. 5 Isosurfaces of the numerical solution for Ψ-equation (5) when a "hand made" label function is used as the initial one. The isosurfaces (their distribution in the vicinity of the island and the main magnetic axis is varied for the sake of clarity) are shown at the toroidal cross sections $\phi = 0, \pi/4, \pi/2, 3\pi/4, \pi$. The solution values at the $\phi = 0$ equatorial plane are also shown.

The procedure for the "b-vector prescription"/"Ψ-equation solution" can be continued using the output label function as an input for the next iteration. If such iterations converge, then the final converged solution Ψ would actually be a label of magnetic surfaces, i.e. a partial numerical solution of the equation $\mathbf{B} \cdot \nabla\Psi = 0$.

It can hardly be expected that such simple iterative process converges. Moreover the problem of magnetic surface label determination is very ill-posed: it has many solutions if it has one. Even in an axisymmetric case with a poloidal flux function as input, the label function keeps changing from iteration to iteration. However the geometry of the label function level lines does not change considerably. The situation gets worse in the case with islands when incorrect "sausage" topology isosurfaces appear.

One of the possible ways to make the problem well-posed is to prescribe an additional flux function which should effectively fix the label function distribution over the magnetic surfaces. In the case of nested flux surfaces, this could be a fixed label function distribution along any ray emerging out of the magnetic axis.

4. Discussion. The correct formulation of the magnetic surface label determination problem is needed in the general case. This is connected with a large freedom in the magnetic field representation if no closure condition is applied to the base vector **b**. Provided with such a formulation, the procedure for magnetic surface label determination in cases with islands can be developed. This procedure is useful by itself as it can provide selective magnetic island mapping with possible regularization for equilibrium magnetic fields. It can be an essential step for the self-consistent 3D equilibrium problem formulation on the basis of scalar functions.

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