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Resistive Equilibrium States of Axisymmetric Plasmas with Compressible Viscous Fluid Flow

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A set of second order partial differential equations is derived describing visco-resistive equilibrium states with poloidal and toroidal flow. This is achieved by a suitable reduction of the full set of the resistive MHD equations to boundary value problems for the rate of expansion and for the poloidal fluxes and toroidal circulations of magnetic induction and flow velocity. Appropriate boundary conditions will be discussed and numerical solutions describing equilibrium states with compressible and viscous fluid flow be presented. This contribution generalizes and extends previous work treating incompressible flow [1], resistive equilibrium states [2] and ideal stationary equilibria [3].

Introduction

The mathematical description of dissipative (resistive and viscous) MHD equilibrium states of an axisymmetric plasma can be grouped in two vectorial and four scalar relations. These are the equation of motion, Ohm's law, the mass continuity and energy balance equations, $\mathrm{div}\mathbf{B} = 0$ and an equation of state provided by some thermodynamic potential. With $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t = 0$ in the plasma region the electric field posesses a scalar potential and a description of the stationary state of the plasma can be formulated in terms of the following quantities: magnetic induction $\mathbf{B}$, flow velocity $\mathbf{v}$. electric potential φ_s, mass density ρ , temperature T and pressure p.
Deferring a consideration of the boundary value problems obtained for temperature, pressure and electric potential and taking into account $\nabla\cdot\mathbf{B} = 0$ and an equation of state, the remaining relations can be put in a form so that they become equations for the kinematic rate of expansion S_M (with $\Delta S_M = \nabla\cdot\mathbf{v}$) and the toroidal circulations and poloidal fluxes of magnetic induction and velocity. Adding the toroidal circulation of the vorticity as an auxiliary quantity a system of six partial differential equations

$$\underline{\Delta}^*\mathbf{F} = \mathbf{G}(\mathcal{D}, \nabla\mathbf{F}, \mathbf{F})$$

is obtained, where $\mathbf{F} = (S_M, \Psi, C, \Psi_M, C_M, C_V)$ comprises expansion rate S_M, poloidal magnetic flux Ψ, toroidal circulation C of $\mathbf{B}$ ($C = \mu_0 J$, where J is the poloidal current), poloidal mass flux Ψ_M and the toroidal circulations of velocity $\mathbf{v}$ and vorticity $\nabla\times\mathbf{v}$, C_M and C_V, respectively. $\underline{\Delta}^*$ is a diagonal matrix of second order partial differential operators and $\mathbf{G}(\mathcal{D}, \nabla\mathbf{F}, \mathbf{F})$ are the corresponding right-hand sides depending on functions $\mathcal{D}$ describing the dissipation.

Theory

For a short explanation how the boundary value problems for $\mathbf{F}$ were obtained we refer to the following physical equations:

$$\nabla\cdot\rho\mathbf{v} = Q_M \tag{1}$$

$$\rho\mathbf{v}\cdot\nabla\mathbf{v} = \nabla\times\mathbf{B}\times\mathbf{B}/\mu_0 - \nabla p - \nabla\cdot\Pi \tag{2}$$

$$\mathbf{v}\times\mathbf{B} = \nabla\phi_s + \mathrm{U}\nabla\phi + \eta\nabla\times\mathbf{B}/\mu_0 \tag{3}$$

Here ϕ_s is the single-valued part of the electric potential and U the loop voltage. Π is the deviatoric part of the pressure tensor. Applying the Helmholtz theorem for decomposing the velocity $\mathbf{v}$ into its irrotational and divergence-free parts and using flux representations for the divergence-free part of $\mathbf{v}$ and for $\mathbf{B}$ we may write

$$\mathbf{v} = \nabla \mathrm{S_M} + \nabla\Psi_\mathrm{M}\times\nabla\phi + \mathrm{C_M}\nabla\phi \tag{4}$$

$$\mathbf{B} = \nabla\Psi\times\nabla\phi + \mathrm{C}\nabla\phi \tag{5}$$

where $\phi = \varphi/(2\pi)$ in right-handed coordinates $(\mathrm{R},\varphi,\mathrm{z})$. For the following considerations we will assume that $\Pi \simeq -2\mu\mathbf{D}$, where μ is the scalar viscosity and $\mathbf{D}$ the deformation tensor. Consider the equations (1), (2) and (3) after the representations (4) and (5) have been inserted. The first one becomes a Poisson equation for the rate of expansion $\mathrm{S_M}$. To the remaining we apply the 3 mutually independent operations $\nabla^2 =: \Delta$, $\nabla\phi\,\cdot$ and $\nabla\phi\cdot\nabla\times$ and focus attention upon the terms containing $\nabla \mathrm{p}$ and $\nabla\phi_s$. The only operation not removing $\nabla \mathrm{p}$ from (2) and $\nabla\phi_s$ from (3) is Δ. The resulting equations containing $\Delta \mathrm{p}$ and $\Delta\phi_s$ will not be considered here. The second and third operations reflect the requirement that the toroidal and poloidal components of (2) and (3) must balance. The fourth order operator appearing due to $\nabla\phi\cdot\nabla\times$ applied to (2) has the form $\Delta^*\Delta^*\Psi_\mathrm{M}$. Splitting the latter into two of second order by introduction of the toroidal circulation of the vorticity $\mathrm{C_V}$ as an auxiliary variable

$$\nabla\times\mathbf{v} = \nabla \mathrm{C_M}\times\nabla\phi + \mathrm{C_V}\nabla\phi \qquad \mathrm{C_V} = \mathrm{R}^2\nabla\times\nabla\Psi_\mathrm{M}\times\nabla\phi = -\Delta^*\Psi_\mathrm{M} \tag{6}$$

the following equations for $\mathrm{S_M}$, Ψ, C, Ψ_M, $\mathrm{C_M}$ and $\mathrm{C_V}$ are obtained:

$$\Delta \mathrm{S_M} = -\frac{1}{\rho}\nabla\rho\cdot\nabla \mathrm{S_M} + \frac{1}{\rho}\mathrm{Q_M} + \frac{1}{2\pi\rho \mathrm{R}}\frac{\partial(\rho,\Psi_\mathrm{M})}{\partial(\mathrm{R,z})} \tag{7}$$

$$\eta\Delta^*\Psi = -\mu_0\left\{\mathrm{U} + \frac{1}{2\pi \mathrm{R}}\frac{\partial(\Psi,\Psi_\mathrm{M})}{\partial(\mathrm{R,z})} - \nabla \mathrm{S_M}\cdot\nabla\Psi\right\} \tag{8}$$

$$\eta\Delta^*\mathrm{C} = -\nabla\eta\cdot\nabla\mathrm{C} - \frac{\mu_0\mathrm{R}}{2\pi}\left\{\frac{\partial(\mathrm{C}/\mathrm{R}^2,\Psi_\mathrm{M})}{\partial(\mathrm{R,z})} - \frac{\partial(\mathrm{C_M}/\mathrm{R}^2,\Psi)}{\partial(\mathrm{R,z})}\right\} + \mu_0\mathrm{C}\Delta\mathrm{S_M} \tag{9}$$

$$\mu\Delta^*\mathrm{C_M} = -\frac{1}{2\pi\mathrm{R}}\left\{\frac{1}{\mu_0}\frac{\partial(\Psi,\mathrm{C})}{\partial(\mathrm{R,z})} - \rho\frac{\partial(\Psi_\mathrm{M},\mathrm{C_M})}{\partial(\mathrm{R,z})}\right\} + \rho\frac{\nabla\mathrm{S_M}\cdot\nabla\mathrm{C_M}}{\mathrm{R}} \tag{10}$$

$$\begin{aligned}\mu\Delta^*\mathrm{C_V} = &-\frac{\mathrm{R}}{2\pi\mu_0}\left\{\frac{\partial((\Delta^*\Psi)/\mathrm{R}^2,\Psi)}{\partial(\mathrm{R,z})} + \frac{\partial(\mathrm{C}/\mathrm{R}^2,\mathrm{C})}{\partial(\mathrm{R,z})}\right\} - \nabla\mu\cdot\nabla\mathrm{C_V} \\ &+\mathrm{C_V}\left\{\mathrm{Q_M} + \frac{1}{2\pi\mathrm{R}}\frac{\partial(\rho,\Psi_\mathrm{M})}{\partial(\mathrm{R,z})}\right\} - 2\pi\mathrm{R}\frac{\partial(\Delta\mathrm{S_M},\mu)}{\partial(\mathrm{R,z})} + \rho\mathrm{R}^2\nabla\mathrm{S_M}\cdot\nabla(\mathrm{C_V}/\mathrm{R}^2) \\ &+\frac{1}{2\pi\mathrm{R}}\left\{\mathrm{C_V}\frac{\partial(\Psi_\mathrm{M},\rho)}{\partial(\mathrm{R,z})} - \mathrm{C_M}\frac{\partial(\mathrm{C_M},\rho)}{\partial(\mathrm{R,z})}\right\} + \frac{\mathrm{R}}{4\pi}\frac{\partial((|\nabla\Psi_\mathrm{M}|^2 + \mathrm{C_M^2})/\mathrm{R}^2,\rho)}{\partial(\mathrm{R,z})} \\ &-\frac{\rho\mathrm{R}}{2\pi}\left\{\frac{\partial(\mathrm{C_V}/\mathrm{R}^2,\Psi_\mathrm{M})}{\partial(\mathrm{R,z})} - \frac{\partial(\mathrm{C_M}/\mathrm{R}^2,\mathrm{C_M})}{\partial(\mathrm{R,z})}\right\}\end{aligned} \tag{11}$$

$$\Delta^*\Psi_\mathrm{M} = -\mathrm{C_V} \tag{12}$$

where $\Delta^*\Psi \equiv R^2\nabla\cdot(R^{-2}\nabla\Psi)$ and $(\partial\alpha,\partial\beta)/\partial(R,z) \equiv 2\pi R(\nabla\beta\times\nabla\alpha)\cdot\nabla\phi$. The dimmed parts in the equations indicate terms which arise due to the nonlinear effects of plasma inertia.

Boundary Conditions. Except for equation (8) which we consider as a free-boundary value problem for the poloidal magnetic flux Ψ describing a magnetically confined plasma in an external conductor system, we assume all conditions to be imposed at a common closed control surface ∂D bounding some three-dimensional domain D. We use the notations $\partial F/\partial s = (\mathbf{t}\cdot\nabla F)|\partial D$ and $\partial F/\partial n = (\mathbf{n}\cdot\nabla F)|\partial D$ for any of the functions F under consideration, where $\mathbf{n}$ is the outward pointing normal of ∂D, s the poloidal arclength along ∂D and $\mathbf{t} = 2\pi R\nabla\phi\times\mathbf{n}$ the tangent vector in clockwise direction.

In discussing (7) for given ρ, Ψ_M, Q_M and Ψ_M boundary conditions for S_M may be of Neumann or Dirichlet type. They are related to the tangent and normal components of the velocity on ∂D by $v_n = \partial S_M/\partial n + (1/(2\pi R))\partial\Psi_M/\partial s$ and $v_t = \partial S_M/\partial s - (1/(2\pi R))\partial\Psi_M/\partial n$. If the plasma is positioned completely inside of D the value of $C = \mu_0 J$ on ∂D is provided by the poloidal current flowing in the toroidal field coils.

For the solution of (10) a boundary value for the toroidal rotation described by C_M in (4) must be prescribed.

Of more involved character are the boundary conditions for the last two equations. As a contribution to the theory of the splitted formulation for problems of the type $\Delta^*\Delta^*\Psi_M = G$ we could find a generalization of the theorem of Quartapelle and Valz-Gris [4] (which refers to plane geometry and the biharmonic type $\nabla^4 F = G$) to axially symmetric problems: A function C_V defined in D is such that $\Delta^*\Psi_M + C_V = 0$ for both $\Psi_M|\partial D$ and $\partial\Psi_M/\partial n$ given, if and only if

$$\iiint\limits_D C_V H_t \frac{d^3x}{R^2} = \iint\limits_{\partial D} \left(\Psi_M \frac{\partial H_t}{\partial n} - H_t \frac{\partial \Psi_M}{\partial n}\right)\frac{dS}{R^2} \tag{13}$$

for any function H_t in D such that $\Delta^* H_t = 0$. From a more practical point of view we have considered continuation and Newton methods for the handling of coupled equations $\Delta^* C_V + G = 0$ and $\Delta^*\Psi_M + C_V = 0$ (originating from $\Delta^*\Delta^*\Psi_M = G$): Prescribe $(\Psi_M|\partial D)(p)$, $(\partial\Psi_M/\partial n)(p)$ and determine $C_V|\partial D$ such that $\partial\Psi_M/\partial n$, which depends on $C_V|\partial D$, equals $(\partial\Psi_M/\partial n)(p)$ (p = prescribed value).

Calculations

Sample calculations for several of the equations (7)-(12) with simplified right-hand sides were performed.

$$\Delta S_M = \frac{1}{\rho} Q_M + \frac{1}{2\pi\rho R}\frac{\partial(\rho,\Psi_{ML})}{\partial(R,z)} \tag{14}$$

was solved for $S_M|\partial D = 0$ by a multigrid finite element method with adaptive refinement [5] (Fig. 1, left) using continuous, up-down symmetric distributions for ρ and Q_M, ρ with a central maximum, Q_M with maxima in the upper and lower half planes. Ψ_{ML} was determined as solution of $\Delta^*\Psi_{ML} + C_{VL} = 0$, where C_{VL} was chosen in the form of an upper and a lower line vortex simulating the vortex distribution in a plasma (see equation (15) below). The poloidal flow pattern with upper and lower stagnation points $\mathbf{v}_p = 0$ (Fig. 1, middle) was obtained by integration of $d(R,z)/dt = (\mathbf{v}\cdot\nabla R, \mathbf{v}\cdot\nabla z) \equiv \mathbf{v}_p$ for some set of initial positions. Another example calculation was carried through solving

$$\Delta^* C_V + \frac{U}{\mu\eta\pi R^2}\frac{\partial\Psi}{\partial z} = 0 \quad \text{and} \quad \Delta^*\Psi_M + C_V = 0 \tag{15}$$

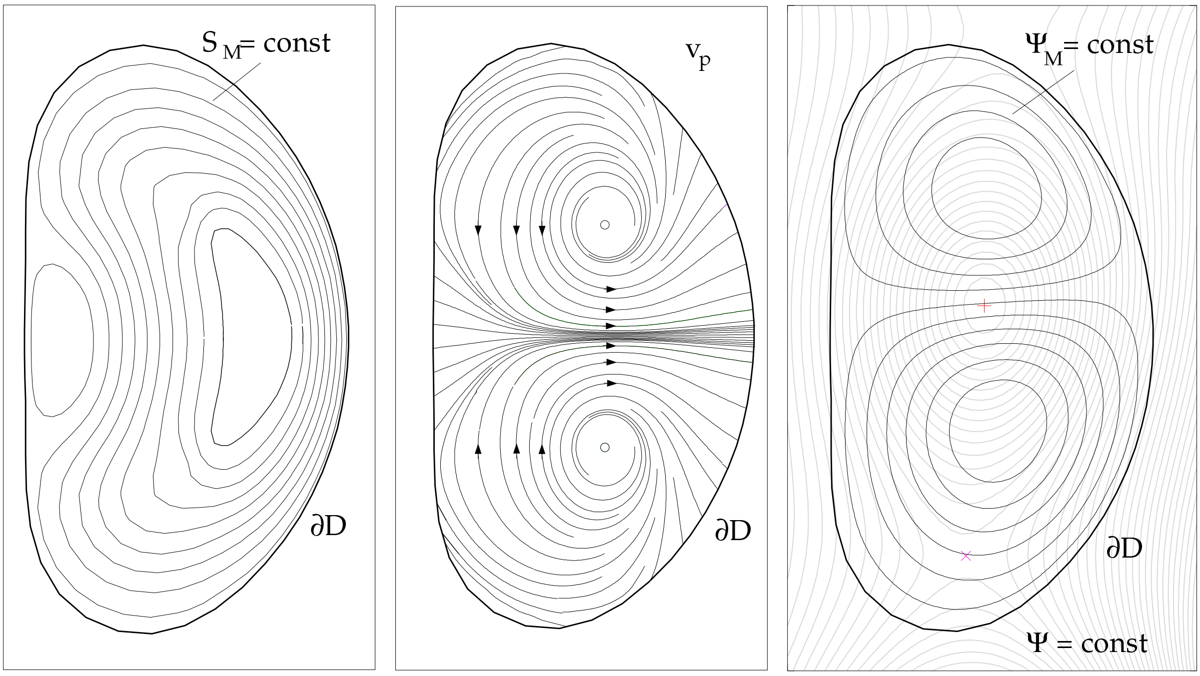

Figure 1: *Left:* Contour plot of the expansion rate S_M as solution of (7). *Middle:* Field lines of poloidal flow with lower and upper stagnation points. *Right:* Flux of mass Ψ_M and poloidal magnetic flux Ψ with magnetic axis and x-point.

for C_V and Ψ_M using for $\Psi(R,z)$ a precalculated, separatrix-defined equilibrium solution, $\eta = \infty$ beyond the separatrix, and μ and η constant in the plasma region. The calculation of C_V and Ψ_M was done with a fixed-boundary Poisson solver under development which is able to handle arbitrarily shaped boundaries and boundary conditions of Dirichlet or Neumann type. (Fig. 1, right) shows the pattern of the poloidal magnetic flux and contours $\Psi_M = \text{const}$. The contours of C_V, not presented here, are similar in form, but have less toroidal outward shift in the R-direction. Note that the extrema of Ψ_M in the lower and upper planes are caused by the extrema of $\partial\Psi/\partial z$ which produce topologically the same distribution Ψ_M as the two line vortices which have been used in the model calculation of the expansion rate S_M.

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