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Calculation of Asymmetric Neoclassical Transport Coefficients by Integration Along the Magnetic Field Line*

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Introduction

The $1/\nu$ neoclassical transport regime is a consequence of the stellarator magnetic field asymmetry. One of the key issues in stellarator optimization is the minimization of neoclassical transport in this regime. In general, this is a huge numerical effort and, therefore, a convenient technique is proposed in the present contribution to calculate the neoclassical transport coefficients. Using this technique, the transport coefficients are expressed in terms of a weighted integral of the geodesic curvature along the magnetic field line. Thus, it takes into account not only the particles trapped within one magnetic field period but also those which are trapped within several magnetic field periods. The method is applied to several stellarator-type configurations and the results are compared to results from Monte-Carlo calculations performed in this context.

Theoretical Background

The starting point is the expression for the particle flux density of trapped particles through the magnetic surface ψ=const of area S (in $w, J_\perp, \sigma$ variables, w is the total energy, $J_\perp = v_\perp^2/B$, $\sigma = \pm 1$)

$$F_n = \frac{\pi}{mS} \oint\limits_{\psi} \frac{\mathrm{d}S}{|\nabla\psi|} B \sum_{\sigma=\pm 1} \int\limits_{e\phi}^{\infty} \mathrm{d}w \int\limits_{J_{\perp\min}^{(\mathrm{abs})}}^{J_{\perp\max}} \mathrm{d}J_\perp \, f_1 \frac{\mathbf{v}\cdot\nabla\psi}{|v_{||}|}. \qquad (1)$$

Here, $v_{||}(\mathbf{r}, w, J_\perp) = \sigma\sqrt{v^2 - J_\perp B}$, $v^2 = 2(w - e\phi)/m$, $J_{\perp\min}^{(\mathrm{abs})} = v^2/B_{\max}^{(\mathrm{abs})}$ is the trapped-passing boundary, $J_{\perp\max}(\mathbf{r}) = v^2/B(\mathbf{r})$, and f_1 is a correction to the Maxwellian distribution $f_0 = f_0(\psi, w)$. Using the explicit expression for the guiding center drift velocity and the property of the equilibrium magnetic field, $(\nabla\times\mathbf{B})\cdot\nabla\psi = 0$, one obtains

$$\frac{\mathbf{v}\cdot\nabla\psi}{|v_{||}|} = \left(\frac{v^2}{|v_{||}|} + |v_{||}|\right)\frac{|\nabla\psi| k_G}{2\omega_c} = -\frac{\partial}{\partial J_\perp}\left(v^2|v_{||}| + \frac{1}{3}|v_{||}|^3\right)\frac{|\nabla\psi| k_G}{B\omega_c}, \qquad (2)$$

with $k_G = (\mathbf{h}\times(\mathbf{h}\cdot\nabla)\mathbf{h})\cdot\nabla\psi/|\nabla\psi|$ the geodesic curvature of the magnetic field line. A partial integration of (1) with respect to $J_\perp$ using the continuity of f_1 and the fact that $f_1 = 0$ for passing particles, leads to

$$F_n = \frac{2\pi}{3mS} \oint\limits_{\psi} \frac{\mathrm{d}S}{|\nabla\psi|} \int\limits_{e\phi}^{\infty} \mathrm{d}w \int\limits_{J_{\perp\min}^{(\mathrm{abs})}}^{J_{\perp\max}} \mathrm{d}J_\perp \, |v_{||}| \left(3v^2 + v_{||}^2\right) \frac{|\nabla\psi| k_G}{\omega_c} \frac{\partial f_1}{\partial J_\perp}. \qquad (3)$$

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Because averages over the volume between two neighboring magnetic surfaces are equivalent to averages along the field lines, this can be written as

$$F_n = \frac{2\pi}{3m} \lim_{L_s\to\infty} \left(\int_0^{L_s} \frac{\mathrm{d}s}{B} |\nabla\psi| \right)^{-1} \int_0^{L_s} \frac{\mathrm{d}s}{B} \int_{e\phi}^{\infty} \mathrm{d}w \int_{J_{\perp\min}^{(\mathrm{abs})}}^{J_{\perp\max}} \mathrm{d}J_\perp \, |v_\parallel| \left(3v^2 + v_\parallel^2\right) \frac{|\nabla\psi| k_G}{\omega_c} \frac{\partial f_1}{\partial J_\perp} \qquad (4)$$

$$= \frac{2\pi}{3m} \lim_{L_s\to\infty} \left(\int_0^{L_s} \frac{\mathrm{d}s}{B} |\nabla\psi| \right)^{-1} \int_{e\phi}^{\infty} \mathrm{d}w \int_{J_{\perp\min}^{(\mathrm{abs})}}^{J_{\perp\max}^{(\mathrm{abs})}} \mathrm{d}J_\perp \sum_{j=1}^{j_{\max}} \int_{s_j^{(\min)}}^{s_j^{(\max)}} \mathrm{d}s |v_\parallel| \left(3v^2 + v_\parallel^2\right) \frac{|\nabla\psi| k_G}{B\omega_c} \frac{\partial f_1^{(j)}}{\partial J_\perp}.$$

Here, $J_{\perp\max}^{(\mathrm{abs})} = v^2/B_{\min}^{(\mathrm{abs})}$ and $B_{\min}^{(\mathrm{abs})}$ is the minimum magnetic field module on the given magnetic surface. In the last expression of (4), the integration along the field line and the velocity space integration have been interchanged. The index j numbers the intervals $[s_j^{(\min)}, s_j^{(\max)}]$ of trapped particle motion with $v^2 - J_\perp B > 0$ for a given $J_\perp$ and w (see Figure 1).

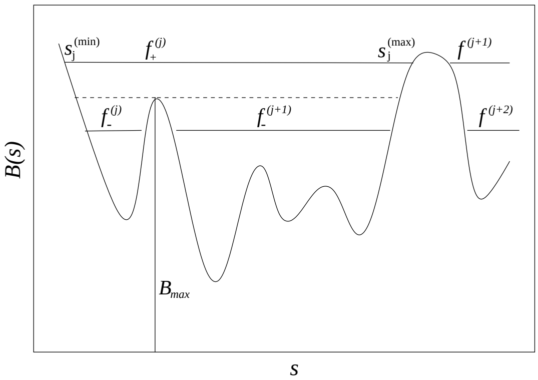

Figure 1. The magnetic field module as a function of the distance along the field line s. The dashed line corresponds to the boundary values of $v^2/J_\perp = B_{\max}$ separating different classes of particles. The turning points $s_j^{\min}$ and $s_j^{\max}$ of trapped particle with $v^2/J_\perp > B_{\max}$ are shown.

The solution of the banana kinetic equation with the Lorenz collision operator is used to evaluate F_n. This solution must satisfy the following conditions at the boundary which corresponds to some local maximum $B_{\max}$ separating different classes of particles trapped within one or several ripple wells,

$$f_-^{(j)} = f_+^{(j)} = f_-^{(j+1)}, \qquad (5)$$

$$\left[I^{(j)} \frac{\partial f^{(j)}}{\partial J_\perp} \right]_+ = \left[I^{(j)} \frac{\partial f^{(j)}}{\partial J_\perp} + I^{(j+1)} \frac{\partial f^{(j+1)}}{\partial J_\perp} \right]_-, \qquad I^{(j)} = 2 \int_{s_j^{(\min)}}^{s_j^{(\max)}} \frac{\mathrm{d}s}{B} |v_\parallel|. \qquad (6)$$

The subscripts $\pm$ indicate classes of trapped particles with $v^2/J_\perp < B_{\max}$ and $v^2/J_\perp > B_{\max}$, respectively. In the $1/\nu$ regime, the banana kinetic equation can be written as

$$\delta\psi^{(j)} \frac{\partial f_0}{\partial \psi} = 4\nu A \frac{\partial}{\partial J_\perp} \left(J_\perp I^{(j)} \frac{\partial f_1^{(j)}}{\partial J_\perp} \right), \qquad (7)$$

where $\nu A(v^2)$ is the pitch-angle scattering frequency. By definition of $\delta\psi^{(j)}$ and with the help of (2) it follows

$$\delta\psi^{(j)} = 2 \int_{s_j^{(\min)}}^{s_j^{(max)}} \frac{\mathrm{d}s}{|v_\parallel|} \mathbf{v} \cdot \nabla\psi = -\frac{1}{3} \frac{\partial H^{(j)}}{\partial J_\perp}, \qquad H^{(j)} = 2 \int_{s_j^{(\min)}}^{s_j^{(max)}} \mathrm{d}s |v_\parallel| (3v^2 + v_\parallel^2) \frac{|\nabla\psi| k_G}{B\omega_c}, \qquad (8)$$

and therefore the derivative of $f_1^{(j)}$ satisfying (6) is obtained from (7),

$$\frac{\partial f_1^{(j)}}{\partial J_\perp} = -\frac{H^{(j)}}{12\nu A J_\perp I^{(j)}}\frac{\partial f_0}{\partial \psi}. \tag{9}$$

Substituting (9) into (4) and introducing instead of $J_\perp$ the new integration variable $b' = v^2/(J_\perp B_0)$ (B_0 is some reference magnetic field) one finally finds,

$$F_n = -\frac{\sqrt{8}}{9\pi^{3/2}}\frac{v_T^2\rho_L^2}{\nu R^2}\epsilon_{\text{eff}}^{3/2}\int\limits_0^\infty \frac{\mathrm{d}z\, e^{-z} z^{5/2}}{A(z)}\frac{n}{f_0}\frac{\partial f_0}{\partial r}, \tag{10}$$

$$\epsilon_{\text{eff}}^{3/2} = \frac{\pi R^2}{8\sqrt{2}}\lim_{L_s\to\infty}\left(\int\limits_0^{L_s}\frac{ds}{B}\right)\left(\int\limits_0^{L_s}\frac{ds}{B}|\nabla\psi|\right)^{-2}\int\limits_{B_{min}^{(\text{abs})}/B_0}^{B_{max}^{(\text{abs})}/B_0}\mathrm{d}b'\sum_{j=1}^{j_{max}}\frac{\hat{H}_j^2}{\hat{I}_j}, \tag{11}$$

$$\hat{H}_j = \frac{1}{b'}\int\limits_{s_j^{(min)}}^{s_j^{(max)}}\frac{\mathrm{d}s}{B}\sqrt{b' - \frac{B}{B_0}}\left(4\frac{B_0}{B} - \frac{1}{b'}\right)|\nabla\psi| k_G, \quad \hat{I}_j = \int\limits_{s_j^{(min)}}^{s_j^{(max)}}\frac{\mathrm{d}s}{B}\sqrt{1 - \frac{B}{B_0 b'}}. \tag{12}$$

Here, $v_T = \sqrt{2T/m}$ is the thermal velocity, $\rho_L = mcv_T/(eB_0)$ is the typical Larmor radius, R is the big radius of the geometrical axis, and $\partial f_0/\partial r$ is the averaged normal derivative. An equation for the energy flux density can be obtained analogously and differs from (10) by factor zT in the sub-integrand. The result (10) differs from the corresponding formula for the standard stellarator (see Eq. (2.16) of [1]) by a simple replacement of the helical modulation amplitude ϵ_h through the effective ripple modulation amplitude ϵ_{eff} given explicitly by (11). The factor $\epsilon_{\text{eff}}^{3/2}$ depends on the magnetic field geometry and naturally takes into account contributions to the $1/\nu$ transport arising from all classes of trapped particles, i.e. particles trapped not only within one magnetic field period but also within several magnetic field periods. For exact quasi-helical symmetry it can be shown that $\hat{H}_j$=0. In order to perform the actual transport calculation, (11) must be supplemented by the magnetic field line equations and the equations for the $\nabla\psi$ calculation (see [2]).

Results

The derived formulae have been applied to study the $1/\nu$ transport of different Helias-type magnetic configurations (original Helias [3] and the quasi-helically symmetric (QHS) stellarator [4]) as well as the U-3M [5] configuration in zero-β limit. In this limit, the magnetic field can be represented in the form of a superposition of toroidal harmonic functions containing the associated Legendre functions. For U-3M a simplified magnetic field with one toroidal harmonic has been considered. The results for $\epsilon_{\text{eff}}^{3/2}$ are shown in Figure 2 and compared to Monte Carlo (MC) calculations of $\epsilon_{\text{effMC}}^{3/2}$ for Helias [3] and U-3M in Figure 3.

It can be seen that the results of the proposed technique are in good agreement with results obtained by MC in the $1/\nu$ regime. The discrepancy of the results for U-3M in the high plasma density region might be explained by the fact that $\epsilon_{\text{eff}}^{3/2}$ corresponds to the transport purely associated with the magnetic field asymmetry whereas the MC calculations take into account also the axisymmetric and helically symmetric parts of the transport and they strongly increase with collision frequency.

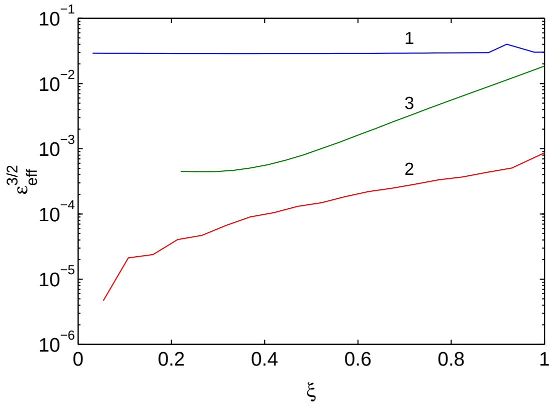

Figure 2. Parameters $\epsilon_{\rm eff}^{3/2}$ for Helias (curve 1), QHS stellarator (curve 2) and U-3M (curve 3); ξ is a dimensionless distance from the magnetic axis (ξ=1 corresponds to the boundaries).

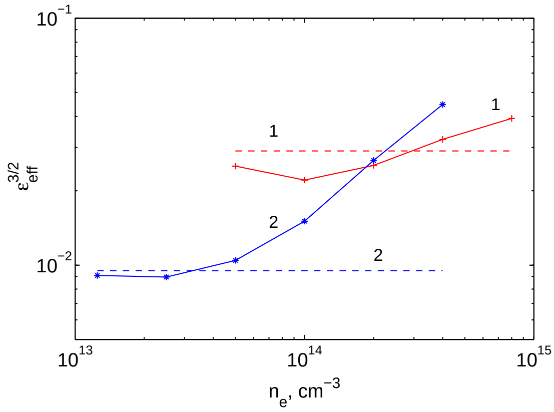

Figure 3. Normalized results of MC calculations (solid lines) for Helias (curves 1) and U-3M (curves 2); dashed lines show the level of $\epsilon_{\rm eff}^{3/2}$ calculated with the technique of integration along the field line.

The results show that in the $1/\nu$ transport regime the neoclassical transport properties for the original Helias [3] magnetic field are only slightly better than those for the standard stellarator (with the same ϵ_h). For the magnetic field of the QHS stellarator [4], the contribution of the magnetic field asymmetry to neoclassical transport is approximately 100 times less than the corresponding contribution in Helias [3].

Summary

Using an analytic solution of the banana kinetic equation in the $1/\nu$ regime, a new formula for the neoclassical transport coefficients is obtained which takes into account all classes of trapped particles. This formula holds in any coordinate system and no simplifying assumptions about the magnetic field are needed. The result is expressed through a set of integrals of the geodesic curvature along the magnetic field lines. The computer time needed to compute the transport coefficients is practically the same as the time needed to compute the magnetic surface of interest through an integration of the magnetic field line equations. In the $1/\nu$ regime, the results of the field line integration technique are in satisfactory agreement with results of Monte-Carlo calculations. For the QHS stellarator magnetic field [4], the $1/\nu$ neoclassical transport coefficients are approximately 100 times less than those of Helias [3].

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