Electromagnetic Wave Beams in an Inhomogeneous Magnetoplasma

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Introduction

The propagation of the electromagnetic waves in inhomogeneous magnetized plasmas has received much careful study. The most well-known and widely used approach is that of the geometrical optics (GO). However, this approach becomes inapplicable when one needs to describe the diffraction effects in the case of the finite width of electromagnetic wave beam. On the basis of the solution given by GO, several approaches have been proposed to describe the diffraction phenomena [1].

In the present work, a generalization is presented of the quasi-optical description of electromagnetic wave propagation to the case of a smoothly inhomogeneous magnetized plasma. Such a description includes the diffraction effects occuring when a wave beam propagates through an essentially anisotropic (in particular, gyrotropic) medium with allowance for curvature and twist of the propagation wave path.

Analysis

We consider the beam propagating through a smoothly inhomogeneous magnetoplasma and consisting of the electromagnetic waves with a prescribed ("ordinary" or "extra-ordinary") polarization. The plasma is assumed to be describable by the Hermitian dielectric tensor $\hat{\varepsilon}$ ($\varepsilon_{ij} = \varepsilon_{ji}^*$). The complex amplitude $\mathbf{E}(\mathbf{r})$ of the electric field of the beam satisfies the following equation:

$$\mathrm{rot}\,\mathrm{rot}\mathbf{E} - k_0^2\hat{\varepsilon}\mathbf{E} = 0, \tag{1}$$

where $k_0 = \omega/c$ is the wavenumber in free space. The time harmonic dependence is chosen as $\exp(-i\omega t)$.

We assume that the width Λ of the beam everywhere along the propagation path, on the one hand, is less than the scale length $L_\varepsilon \sim |\varepsilon_{ij}/(\nabla \varepsilon_{ij})|$ and, on the other hand, is greater than the local wavelength λ ($\lambda \ll \Lambda \ll L_\varepsilon$).

Thus, there are two small parameters in our problem: $\nu = \lambda/\Lambda$ and $\mu = \Lambda/L_\varepsilon$. When the conditions $\nu \ll 1$ and $\mu \ll 1$ are satisfied, the wave beam is localized in the vicinity of the certain GO ray $\mathbf{r}_0$ (we shall call this ray as a reference ray). The equations for the reference ray are of the form [2]:

$$\frac{d\mathbf{r}_0}{d\tau} = \mathbf{p}_0 - \frac{1}{2}\frac{\partial n^2}{\partial \mathbf{p}_0} = S_0\mathbf{s}_0, \quad \frac{d\mathbf{p}_0}{d\tau} = \frac{1}{2}\frac{\partial n^2}{\partial \mathbf{r}_0}, \tag{2}$$

where $\mathbf{p}_0(\tau)$ is the wave vector normalized to the k_0, S_0 is the module of the ray vector, and $n \equiv n(\mathbf{r}, \mathbf{p})$ is the refractive index of one of the characteristic modes.

Regarding the reference ray as known and its equation $\mathbf{r} = \mathbf{r}_0(\tau)$ as given, we introduce curved orthogonal coordinates $\xi^0 = \tau$, ξ^1, ξ^2 (see, e.g., [3]) by

$$\mathbf{r} = \mathbf{r}_0(\tau) + \xi^1\,\mathbf{e}_1(\tau) + \xi^2\,\mathbf{e}_2(\tau), \tag{3}$$

where

$$
\begin{aligned}
\mathbf{e}_1(\tau) &= \mathbf{n}\,\cos\vartheta - \mathbf{m}\,\sin\vartheta, \\
\mathbf{e}_2(\tau) &= \mathbf{n}\,\sin\vartheta + \mathbf{m}\,\cos\vartheta, \\
\frac{d\vartheta}{d\tau} &= T\,S_0,
\end{aligned}
$$

$\mathbf{n}$ and $\mathbf{m}$ are the unit vectors normal and binormal to the reference ray, and T is the twist of the reference ray. The metric tensor g_{ij} in the coordinate system is given by

$$g_{00} = h_\tau^2; \qquad g_{ij} = \delta_{ij}; \quad i,j = 1,2; \qquad g_{0i} = 0, \quad i \neq 0; \tag{4}$$
$$h_\tau = S_0\left(1 - K\left(\xi_1\cos\vartheta + \xi_2\sin\vartheta\right)\right),$$

where K is the curvature of the reference ray.

Note that the turning of the orthonormalized basis $\{\mathbf{s}_0, \mathbf{e}_1, \mathbf{e}_2\}$ by the angle $\vartheta(\tau)$ corresponds to the rotation of the polarization ellips, which occurs when a wave propagates through an inhomogeneous gyrotropic medium.

In view of the above, we shall seek solution of the equation (1) in the form of asymptotic series in the parameters μ and ν:

$$\mathbf{E} = \sum_{k=0}^{\infty} \mathbf{E}_k(\tau, \xi_1, \xi_2)e^{ik_0(\varphi_0 + q_1\xi_1 + q_2\xi_2)}, \tag{5}$$

where

$$q_i = -\frac{1}{2}\frac{\partial n^2}{\partial p_{0i}}\,|_0, \quad p_{0i} = \mathbf{p}_0 \cdot \mathbf{e}_i, \quad i = 1,2,$$
$$\frac{1}{S_0^2}\left(\frac{d\varphi_0}{d\tau}\right)^2 + q_1^2 + q_2^2 = n_0^2,$$
$$\mathbf{p}_0 = \nabla\varphi_0;$$

$\mathbf{E}_0(\tau, \xi_1, \xi_2) = A(\tau, \xi_1, \xi_2)\mathbf{u}_0(\tau)$; $\mathbf{u}_0(\tau)$ is the complex polarization vector on the reference ray; $|\mathbf{E}_{k+1}/\mathbf{E}_k| \sim \mu, \nu$. The subscript "0" denotes that the corresponding functions are taken at $\mathbf{r} = \mathbf{r}_0(\tau)$ and $\mathbf{p} = \mathbf{p}_0(\tau)$.

It can be shown that the following relations are satisfied:

$$
\begin{aligned}
K\,\cos\vartheta\,\frac{d\varphi_0}{d\tau} + \frac{dq_1}{d\tau} &= \frac{1}{2}\frac{\partial n^2}{\partial \xi_1}\,|_0, \\
K\,\sin\vartheta\,\frac{d\varphi_0}{d\tau} + \frac{dq_2}{d\tau} &= \frac{1}{2}\frac{\partial n^2}{\partial \xi_2}\,|_0 \ .
\end{aligned}
\tag{6}
$$

On substituting (5) into (1) and equating the terms of the same orders of μ and ν, we get:

$$
\begin{aligned}
\hat{Q}\mathbf{E}_0 &= 0, \tag{7} \\
\hat{Q}\mathbf{E}_1 &= \hat{F}\mathbf{E}_0, \tag{8} \\
\hat{Q}\mathbf{E}_2 &= \hat{L}\mathbf{E}_0 + \hat{F}\mathbf{E}_1 \tag{9}
\end{aligned}
$$

where $\hat{Q}\mathbf{E}_k = p_0^2\mathbf{E}_k - \mathbf{p}_0(\mathbf{p}_0\mathbf{E}_k) - \hat{\varepsilon}_0\mathbf{E}_k$; $\hat{F}$ and $\hat{L}$ are certain differential operators. Here we limit ourselves to consideration only of the second order terms.

Equation (7) is the usual dispersion relation which determines the eikonal in geometric optics.

The solvability condition of the eq. (8) is as follows:

$$\mathbf{u}_0^*\hat{F}\mathbf{E}_0 = 0. \tag{10}$$

With the help of (6), one immediately comes to the relation (10) which gives no useful result. Therefore, we have to deal with eq. (9). The solvability condition of eq. (9) is

$$\mathbf{u}_0^*\hat{L}\mathbf{E}_0 + \mathbf{u}_0^*\hat{F}\hat{Q}^{-1}\hat{F}\mathbf{E}_0 = 0. \tag{11}$$

Using (6) and (11), we can obtain the following quasi-optical equation for the complex amplitude $A(\tau, \xi_1, \xi_2)$:

$$2ik_0\frac{\partial A}{\partial \tau} + T_{ij}(\tau)\frac{\partial^2 A}{\partial \xi_i \partial \xi_j} + 2ik_0\left(B_{ij}(\tau)\xi_j\frac{\partial A}{\partial \xi_i} + \frac{1}{2}B_{ij}(\tau)\delta_{ij}A\right) - k_0^2\Phi_{ij}(\tau)\,\xi_i\xi_j A = 0. \tag{12}$$

Here, $i, j = 1, 2$; the quantities $T_{ij}(\tau)$, $B_{ij}(\tau)$ and $\Phi_{ij}(\tau)$ depend on the values of the refractive index $n(\mathbf{r}, \mathbf{p})$ and its derivatives on the reference ray:

$$
\begin{aligned}
T_{ij} &= \delta_{ij} - \frac{1}{2}\frac{\partial^2 n^2}{\partial p_i \partial p_j}, \\
B_{ij} &= -\left(\frac{\partial^2 n^2}{\partial p_i \partial \xi_j} + \frac{1}{S_0}\frac{\partial^2 n^2}{\partial p_i \partial p_s}\frac{\partial n^2}{\partial \xi_j}\right), \\
\Phi_{ij} &= -\left(\frac{1}{2}\frac{\partial^2 n^2}{\partial \xi_i \partial \xi_j} + \frac{1}{4S_0^2}\left(1 - \frac{1}{2}\frac{\partial^2 n^2}{\partial p_s^2}\right)\frac{\partial n^2}{\partial \xi_i}\frac{\partial n^2}{\partial \xi_j} + \frac{1}{\dot{\varphi}_0}\frac{\partial n^2}{\partial \xi_i}\left(\frac{1}{2S_0}\frac{\partial n^2}{\partial \xi_j} + \dot{q}_j\right)\right).
\end{aligned} \tag{13}
$$

The solution of the Cauchy's problem for the equation (12) ($A\mid_{\tau=0} = W(\xi_1, \xi_2)$) is given by

$$A(\tau, \xi_1, \xi_2) = \frac{k_0^2}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{W}(\kappa_1, \kappa_2)D\,\exp(ik_0\Psi(\xi_1, \xi_2; \kappa_1, \kappa_2))d\kappa_1 d\kappa_2, \tag{14}$$

where

$$
\begin{aligned}
\tilde{W}(\kappa_1, \kappa_2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}W(\xi_1, \xi_2)\exp(-ik_0(\xi_1\kappa_1 + \xi_2\kappa_2))d\xi_1 d\xi_2, \\
\Psi &= \alpha_{ij}\xi_i\xi_j + \beta_{ij}\xi_i\kappa_j + \gamma_{ij}\kappa_i\kappa_j \quad (i, j = 1, 2).
\end{aligned}
$$

The parameters α_{ij}, β_{ij}, γ_{ij}, and D satisfy the following system of ordinary differential equations:

$$
\begin{aligned}
2\dot{\alpha}_{ij} + 2B_{kj}\alpha_{ki} + T_{kn}\alpha_{ki}\alpha_{jn} + \Phi_{ij} &= 0, \\
2\dot{\beta}_{ij} + 2B_{ki}\beta_{kj} + T_{kn}(\alpha_{in}\beta_{kj} + \alpha_{ki}\beta_{jn}) &= 0, \\
2\dot{\gamma}_{ij} + T_{kn}\beta_{ki}\beta_{jn} &= 0, \\
2\dot{D} + DT_{ij}\alpha_{ij} + DB_{ij}\delta_{ij} &= 0.
\end{aligned} \tag{15}
$$

The initial data (with $\tau = 0$) are $\alpha_{ij}(0) = 0$, $\beta_{ij}(0) = \delta_{ij}$, $\gamma_{ij}(0) = 0$, and $D(0) = 1$.

Conclusion

We have derived the quasi-optical equation for the case of an inhomogeneous magnetized plasma. The problem of finding the solution of the quasi-optical equation have been reduced to solving the system of ordinary differential equations and calculating the integral convolutions. These ordinary differential equations do not depend on concrete structure of the wave pattern but enable us to describe universal focusing and defocusing properties of the beam trajectory.

Note that the method proposed in the present paper essentially differs from the well-known generalizations of the geometrical optics, in which a wave field is found by the corresponding ray approach.

Unlike [1], our method enable us to calculate the fields in any transverse cross sections of wave beams with arbitrary initial distributions of amplitude, phase and polarization. On the basis of the quasi-optical solution obtained, a numerical code for readily calculating the wave fields in fusion devices can be elaborated.

Acknowledgments

References

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