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Structural stability of tokamak equations: confinement transitions.

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A study of the structural stability of the Grad-Shafranov equation is presented. It is shown how confinement transitions and the formation of transport barriers may be associated with the existence of multiple solutions of the equilibrium equation.

Let us first define structural stability and criticality of an equation. Given an implicit equation for the function u, parameterized by λ:

$$G(u, \lambda)= 0, \tag{1}$$

the equation is said to be structurally unstable (or critical) if a small perturbation <u>of the equation</u> (for instance, the parameter λ) leads to an essential change in its solution. Essential changes are changes in existence and multiplicity of solutions. A solution u of a critical equation is called a critical solution.

It is known [1] that structural stability is associated with transversality of the operator G(u) as it crosses the G=0 plane. Structural <u>instability</u> is associated with a zero slope of G at the critical solution. A suitably general definition of the slope of an operator is given by its linearization near a trial function u*:

$$G_L(\tilde{u})\Big|_{u*} \equiv \lim_{\varepsilon \to 0} \frac{G(u*+\varepsilon\,\tilde{u}) - G(u*)}{\varepsilon} \equiv \tilde{u}\,\frac{\partial G}{\partial u}\Big|_{u*} \equiv \tilde{u}\, G_u(u*) \tag{2}$$

The criticality condition for equation (1) is given by the existence of a function $u\neq 0$ such that $G(u)= 0$ and $G_u(u)= 0$.

Simple polynomial examples of criticality are discussed in [1]. They can be generalized to more complicated equations [2]. Here it will suffice to consider the first two non-linearities, analogous to 2nd and 3rd order equations. If an equation is critical and in a neighborhood of the critical point (including it) the operator G(u) has non-zero concavity, the equation is locally equivalent to a 2nd order equation: it can have 2 distinct solutions, one critical solution, or no solution ("fold catastrophe"). If the concavity of G crosses zero at the critical solution, ($G_{uu}(u)=0$), equation (1) may be locally equivalent to a cubic equation ("cusp catastrophe"), which always has at least one solution, and may have up to 3 distinct solutions.

Let us consider the Grad-Shafranov equation, which describes MHD equilibrium of toroidally (or helically) symmetric plasma. The equation is

$$\begin{aligned} G(\Psi) &= \frac{1}{\mu_0 R}\left(\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial \Psi}{\partial R} + \frac{\partial^2 \Psi}{\partial Z^2}\right) + \left(R\, p' + \frac{(F^2)'}{2\mu_0 R}\right) = 0 \\ &= L(\Psi) + J(\Psi) \end{aligned} \tag{3}$$

Here, cylindrical coordinates (R,Z,ζ) are used; the magnetic field is described as $\vec{B} = F\nabla\zeta + \nabla\Psi \times \nabla\zeta$; Ψ is the poloidal magnetic flux per radian, measured from the plasma magnetic axis; the prime indicates derivative with respect to Ψ; p is the plasma pressure; F is the poloidal flux density, related to the poloidal current density in the plasma by $\vec{j}_{pol} = -F'\vec{B}_{pol}$. When $\vec{j}_{pol}$ adds to (is opposed to) the externally driven toroidal field, the plasma is said to be paramagnetic (diamagnetic). L is a linear operator acting on Ψ, p and F are non-linear functions of Ψ only, and J is the toroidal current density.

The linearization of (3) is given by:

$$G_L(\Psi)\big|_{\Psi^*} = L(\Psi) + \Psi\, J_\Psi(\Psi^*) \qquad (4)$$

If Ψ^* is a solution of (3), $L(\Psi^*) = -J(\Psi^*)$, and if it is a critical solution, in the limit $\Psi \to \Psi^*$ the slope is zero. This translates into the following criticality condition:

$$\Psi^*\, J_\Psi(\Psi^*) = J(\Psi^*) \qquad (5)$$

In tokamak plasmas J is usually positive inside the plasma, so criticality can only be present in equilibria that have an increasing current density as a function of Ψ: i.e., there must be a minimum of J in the plasma. In the neighborhood of such a minimum, the equation undergoes a catastrophe. If at the critical point $J_{\Psi\Psi}>0$ (<0) we have a fold catastrophe: an increase (decrease) in J leads to a loss of solution. If $J_{\Psi\Psi}=0$ we have a cusp: a new pair of solutions may appear. We also should recall that the plasma boundary shape is affected by the sign of J_Ψ at an X-point [3].

Local criticality, when equation (5) is verified at a point in the plasma, is sufficient for loss of solution: the plasma can no longer be described as ideal (non-resistive), in equilibrium, and toroidally symmetric. The appearance of an MHD mode, or of magnetic islands, or the occurrence of a disruption, local or global, are the possible states the plasma may change to. Equation (5) provides a very simple test of MHD: if (5) is verified <u>and</u> $J_{\Psi\Psi}\neq 0$, we have a fold catastrophe and the equilibrium is unstable. Note that usual MHD tests are based on linearizing δW: the criticality condition corresponds to $\delta W_L=0$, when the linear approximation is not valid.

Global criticality, present when equation (5) is verified in a closed poloidal loop around the magnetic axis, allows the appearance of a new solution: there can be more than one function $\Psi(R,Z)$ that solve the same equation. Once the plasma is near a globally critical equilibrium, a small perturbation of p' or $F^{2\prime}$ leads to one or another solution. The solution with higher ∇p is more diamagnetic. The difference between the two equilibria can be understood as a difference in the angle between $\vec{j}$ and $\vec{B}$, since $\vec{j} \times \vec{B}$ supports ∇p. A change in angle can happen in a fast time scale.

Local criticality at a rational surface may be sufficient for the appearance of a new solution in a given flux bundle of the surface, transiently breaking symmetry.

The evolution of the plasma equilibrium is controlled by the plasma transport equations (particle, heat and current diffusion). Once the transport equations have taken the equilibrium to a critical point, a high pressure gradient solution is favored if $\dot{p}_p > 0$ (i.e., the greater the pressure, the greater its rate of growth). Any one of the transport equations can be responsible for driving the equilibrium towards the higher ∇p solution, but from then on $\nabla\Psi$ changes, affecting all the profiles (in a possibly complex manner). Contributing factors in the $\dot{p}_p > 0$ direction are present in non-linearities of the plasma transport equations, as we discuss in the following:

1. In the particle transport equation, the ionization source is proportional to the electron impact ionization rate coefficient, which exhibits a maximum near ~100 eV. When the plasma edge temperature rises above this critical T_e value, a further increase in temperature results in a reduced local density, and further penetration of the neutrals into the plasma, which raises the local density further inwards. This may result in an increasing ∇n, contributing to a positive $\dot{p}_p$.

2. In the plasma heat equation, low Z impurities that typically radiate at the plasma edge introduce a maximum in the collisional radiative power loss emission coefficients near ~100 eV. Above that critical T_e, an increase in edge electron temperature leads to a decrease in the sink term, and therefore to a higher T_e, possibly contributing to a positive $\dot{p}_p$ (thermal bifurcation discussed in [4]). Typically the edge is heated ohmically, with a heat source term Q= $\eta\,(J - J_{bs})^2$, (J_{bs} is the bootstrap current); because the plasma resistivity η is proportional to $T_e^{-3/2}$, low edge temperatures lead to high resistivity and low (and decreasing) edge current, opposing the approach of the equilibrium to the critical point. High edge temperatures are therefore necessary for edge criticality. Also, note that a current profile with a minimum of $\left|J - J_{bs}\right|$ in it can heat the edge more effectively, raising the edge temperature, and self-consistently driving towards a hot edge plasma solution.

3. Collisional transport coefficients are decreasing functions of temperature, so at higher temperatures confinement improves, contributing to a positive $\dot{p}_p$ [3].

4. As the equilibrium develops a steeper $\nabla\Psi$ profile, the local radial electric field increases (since the electric potential is a flux function and the flux surfaces are "bunching up"). An increase in radial electric field and/or its shear is known to lead to a reduction in turbulence-driven transport. We find the origin of the sudden change in E_r to be in the criticality of the equilibrium equation, rather than in the heat equation.

All of the above factors lead us to construct a model for the L to H transition, in which plasma edge heating and absence of high Z impurities allow the formation of a minimum in J, leading to dual solutions for the equilibrium equation. The high ∇p solution, with increased E_r, higher T, and low transport, is preferred for a sufficiently hot diamagnetic edge. As the high gradient region grows into the plasma, the position of the J minimum moves inwards. The equilibrium remains vulnerable to a fold catastrophe if J at the minimum rises too fast, or if a maximum of J forms inside the plasma edge: this might explain ELM's. Another reason for the more or less periodic collapse of the edge plasma may be found in the heat transport equation: a hot edge has low collisionality, and the fraction J_{bs}/J increases as the edge heats up, diminishing ohmic heating. The thermal balance may collapse and drive the plasma back to a state with a colder edge.

A positive plasma current ramp drives parallel current more efficiently (perpendicular resistivity is higher than parallel), increasing paramagnetism in the edge plasma. If the edge equilibrium is near criticality, but still paramagnetic, a negative current ramp can drive it towards the higher ∇p solution (more diamagnetic).

The equilibrium criticality condition may also be invoked to explain the formation of internal transport barriers. The criticality condition (5) needs to be satisfied for a high ∇p solution to appear, either from a fold ($J_{\Psi\Psi}\neq 0$) or (more likely) from a cusp ($J_{\Psi\Psi}=0$) catastrophe. The evolution of the equilibrium is then controlled by the non-linear dependence of heat and particle deposition profiles an transport coefficients on temperatures, density and electric fields.

A study of the structural stability of the steady-state solutions of the transport equations can also illuminate aspects of plasma behavior (describing conditions for bifurcations to be possible), but for the moment its is easier to comment on the effect of different terms on equilibrium evolution, as we did above.

In conclusion, the study of the structural stability of the plasma equilibrium can tie together a large variety of experimental observations that have up until now been very difficult to explain. Much more can be learned when the coupled set of plasma equations is treated consistently, but this must await numerical simulation.

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