

## Stability of localized structures in magnetized plasmas

J. Bergmans, B. N. Kuvshinov, V. P. Lakhin, T. J. Schep

*FOM-Instituut voor Plasmafysica "Rijnhuizen", Association Euratom-FOM, TEC,  
Postbus 1207, 3430 BE Nieuwegein, The Netherlands*

### Abstract

The two-fluid equations for a magnetized plasma admit solutions in the form of drift-Alfvén vortices. These localized structures are analogues of vortices in ordinary fluids. In this work we investigate the linear stability of regular configurations of these vortices.

The plasma model describes drift-Alfvén perturbations of a plasma in a strong background magnetic field. The equations describe three pointwise conserved scalar fields, or generalized vorticities, advected by streaming potentials which are linear combinations of the magnetic flux and the electrostatic potential. Equations with similar structure also arise in hydrodynamics (the Euler equation) and drift-wave theory, electron magnetohydrodynamics, and geostrophic fluid dynamics. These equations are limiting cases of the Alfvén equations used here.

All these models allow singular distributions of the vorticity fields. These point-vortices or filaments move like interacting particles and their motion is Hamiltonian. Filaments can form various regular equilibria: single rows, double rows, and single and double rings. It is found that their stability is quite different in the various models. Most notably, rings which are unstable in the Euler and geostrophic cases are stable in the Alfvén case for a range of values of the ring radius. On the other hand, for rings consisting of a small number of filaments, unstable modes are found which were stable in the Euler and geostrophic cases.

### Drift-Alfvén filaments

The two-fluid drift-Alfvén model for a magnetized plasma has exact solutions in the form of singular current-vortex filaments [1]. This two-fluid plasma model has been applied to investigate integrable filament dynamics and collapses [2]. An extensive study of the spectral stability of equilibrium configurations of Alfvén filaments (infinite rows, double rows, and single and double rings) has been made in [3]. This contribution gives a short introduction as well as some results of the stability analysis of rings of filaments.

The model describes low-frequency plasma phenomena with spatial scales that may range from MHD lengths to the electron collisionless skin depth. It incorporates electron inertia and finite ion-sound gyroradius effects. The electron and ion fluids are embedded in a strong magnetic field in the  $z$ -direction, and the plasma pressure is assumed to be small, so that the magnetic and electric fields can be described by

$$\mathbf{B} = B_0 \mathbf{e}_z + \mathbf{e}_z \times \nabla \psi, \quad \mathbf{E} = -\nabla \varphi + \mathbf{e}_z \partial_t \psi. \quad (1)$$

Here  $\psi$  is the magnetic flux function and  $\varphi$  the electrostatic potential. Solutions are assumed to travel in the  $z$ -direction (along the main magnetic field) with a constant velocity  $u_z = \lambda^{-1} v_A$ .

The perpendicular dynamics are described by the conservative advection of three scalar vorticity fields, each by its own, incompressible, velocity field:

$$\frac{\partial G_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \nabla G_\alpha = 0, \quad \alpha = +, -, 0. \quad (2)$$

The fields are given by

$$G_\pm = (\psi - \lambda_e^2 J) - \lambda(\varphi \pm \lambda_e J) - (\pm \lambda_e - \lambda) \ln n, \quad G_0 = \ln n - \nabla^2 \varphi. \quad (3)$$

The velocity fields to match are

$$\mathbf{v}_\pm = \mathbf{e}_z \times \nabla \left( \frac{d_e \varphi \mp \psi}{d_e \mp \lambda} \right), \quad \mathbf{v}_0 = \mathbf{e}_z \times \nabla \varphi. \quad (4)$$

All lengths are normalized to the ion sound gyroradius  $\rho_s$ , times to the ion gyrofrequency,  $d_e$  is the normalized electron inertial skindepth,  $J = \nabla^2 \psi$  is the parallel current, and  $n$  is the particle density.

Solutions of (2) are characterized by the inverse scale length  $\gamma$ , defined by  $\gamma^2 = (1 - \lambda^2)/(d_e^2 - \lambda^2)$ . When  $\gamma^2$  is negative the solutions are drift-Alfvén waves, and if  $\gamma^2 \geq 0$ , the solutions are localized, and we obtain drift-Alfvén *vortices*. A special case of these localized structures is when the fields  $G_\alpha$  are distributed in singular *filaments* of the form

$$G_\alpha(\mathbf{r}) = \sum_i \kappa_{\alpha,i} \delta(\mathbf{r} - \mathbf{r}_{\alpha,i}). \quad (5)$$

Here,  $\mathbf{r} = (x, y)$ ,  $\mathbf{r}_{\alpha,i}$  is the position and  $\kappa_{\alpha,i}$  the intensity (strength) of filament  $i$  which is of type  $\alpha$ . Filaments move like discrete particles in the velocity field of all other filaments. The motion is Hamiltonian, and is described by

$$\frac{dx_n}{dt} = \frac{1}{\kappa_n} \frac{\partial H}{\partial y_n}, \quad \frac{dy_n}{dt} = -\frac{1}{\kappa_n} \frac{\partial H}{\partial x_n}. \quad (6)$$

The Hamiltonian is given by

$$H = -\frac{1}{4\pi(1 - \lambda^2)} \sum'_{i,j} \kappa_i \kappa_j [\ln(r_{ij}) + c_i c_j K_0(\gamma r_{ij})]. \quad (7)$$

The prime denotes exclusion of the  $i = j$  terms. Note that the Hamiltonian only depends on the distance between vortices  $r_{ij}$ . The interaction coefficients  $c_\alpha$  are given by  $c_\pm = (1 \mp d_e \lambda)/(\pm d_e - \lambda)$  and  $c_0 = -\lambda$ .

These solutions are exactly analogous to the well known point vortex solutions of ideal hydrodynamics, which move according to the Hamiltonian  $H = -(4\pi)^{-1} \sum' \kappa_i \kappa_j \ln r_{ij}$ . This logarithmic interaction is long ranged and scale invariant. The Hamiltonian (7) reduces to that of hydrodynamic (Euler) point vortices in the limit  $d_e, \lambda \rightarrow 0$ .

Another well known fluid model which has point vortex solutions is the geostrophic equation, which has a short ranged, screened interaction described by a modified Bessel function,  $H = (4\pi)^{-1} \sum' \kappa_i \kappa_j K_0(r_{ij})$ . The geostrophic interaction can also be obtained as a limiting case from (7) when we take  $d_e \rightarrow 0, \lambda \rightarrow \infty$ .

The Hamiltonian (7) shows that the interaction between drift-Alfvén filaments is a combination of the long ranged hydrodynamic and the screened geostrophic interactions. The relative importance of the two contributions is determined by the coefficients  $c_\alpha$  and the Bessel length scale  $\gamma^{-1}$ . This mixed character of the interaction turns out to have a profound influence on the stability and the dynamics of a system of plasma filaments.

## Filament rings

Consider a ring consisting of  $N$  identical vortices ( $c_i = c = C_+, C_-,$  or  $C_0$  for all  $i$ ) placed at the vertices of a regular  $N$ -sided polygon. There is a central vortex of relative strength  $\kappa_0$ . For simplicity we take the central filament of the same type as the ring vortices. This configuration rotates steadily and to analyze its stability we linearize the  $2(N + 1)$  equations of motion (6) around these positions. This results in a set of linear ordinary differential equations for the perturbation. We perform the usual exponential stability analysis by writing the perturbation as a sum of eigensolutions, each having a time behavior  $\exp(\lambda_i t)$ , with  $i = 1, \dots, 2(N + 1)$ . Whenever there is an eigenvalue  $\lambda_i$  with a positive real part, the ring is unstable and the growth rate is  $\lambda_i$ . Because for a single ring the eigenvalues behave like  $\sim R^{-2}$  for small radii, all growth rates will be multiplied by  $R^2$ . For brevity we will only compare the result with the known results for hydrodynamic point vortex rings [4]. An extensive study of geostrophic rings can be found in [5], and the full comparison between the three types of vortices is made in [3].

**Rings without a central filament** In the absence of a central filament the stability of the ring is determined by the eigenvalues of the  $2N \times 2N$  coefficient matrix. For the Euler case the growth rates are independent of  $R$ . Rings with  $N < 7$  are stable, and rings with  $N > 7$  are unstable. The heptagon,  $N = 7$ , is neutrally stable, i.e. its growth rate is equal to zero.

When the ring consists of Alfvén filaments the growth rates do depend on the radius of the ring. Figure 1 shows that the stability of the Alfvén ring is qualitatively different from the Euler case. Rings with  $N = 4, 5, 6, 7$  are unstable in a finite range of the ring radius, here the mixed logarithmic-Bessel character of the interaction introduces an instability. The two terms in the interaction can also balance each other, as can be seen for  $N = 8, 9, 10$  (and also for higher  $N$ ) where the ring is stable for an interval of  $R$  values. For large radius the interaction becomes purely logarithmic and the growth rates tend to their Euler values.

**Rings with a central filament** We consider the full  $2(N + 1) \times 2(N + 1)$  system of equations. For hydrodynamical vortex rings, for every number of vortices, the central vortex stabilizes the configuration for a range of values of its relative strength  $\kappa_0$ . This is shown in Fig. 2(Left) for several different numbers of ring vortices  $N$ . Again, the hydrodynamic growth rates do not depend on the ring radius.

When we look at the case when the central filament has the same strength as the ring filaments,  $\kappa_0 = 1$ , hydrodynamic rings are unstable for  $N = 2$ . A ring of three vortices is neutrally stable,  $N = 4, 5, 6, 7$ , and 8 are stable, and rings with more than eight vortices are unstable.

Comparing this with the results shown in Fig. 2(Right), the effect of a central filament on Alfvén filament rings is demonstrated. In the Alfvén case the unstable motion of the central filament for  $N = 2$  and  $N = 3$  is stabilized by the mixed logarithmic-Bessel interaction in an interval of  $R$  values. Two disjoint unstable regions exist for both  $N = 4$  and  $N = 5$  rings and

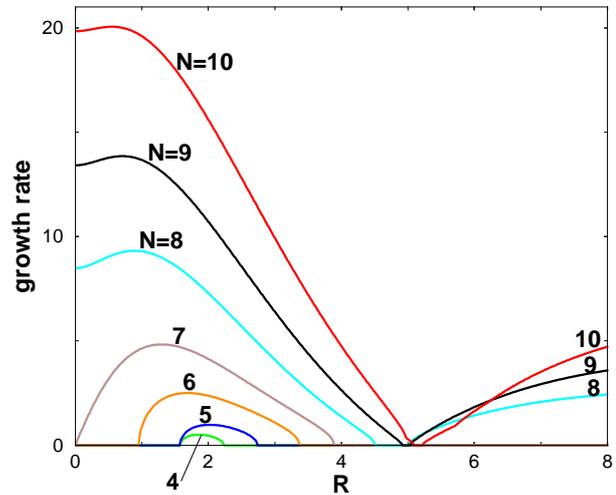


Figure 1: Growth rates for rings of Alfvén filaments without a central filament against the ring radius  $R$  for different numbers of filaments,  $c = 2.0$ .

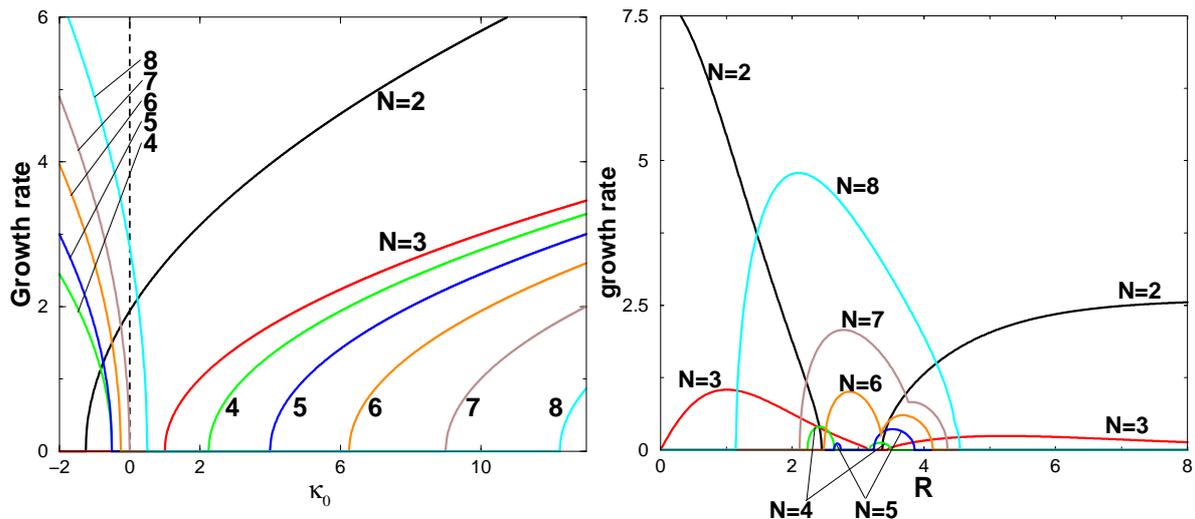


Figure 2: Left: Growth rate versus central vortex strength  $\kappa_0$  for hydrodynamic rings. Right: Growth rates against radius for rings of Alfvén filaments with a central filament of relative strength  $\kappa_0 = 1$ .

also the other rings which are shown ( $N = 6, 7, 8$ ) are unstable when the radius lies in a range where the mixed interaction manifests itself.

To get a complete picture of the stability of a ring of Alfvén filaments as a function of its radius and the strength of the central filament, we give one example for  $N = 5$  in Fig. 3.

**Final remarks** We are now extending the work on point vortices to the dynamics of finite-size vortex patches. To this end we use a contour dynamics calculation which has been modified to solve the three-field motion given by (2). Also vortex sheets will be considered, where the vorticity is distributed along a line or a thin strip. This situation is analogous to the Kelvin–Helmholtz instability in the Euler equation, when the fluid velocity has a jump across a (thin) shear layer.

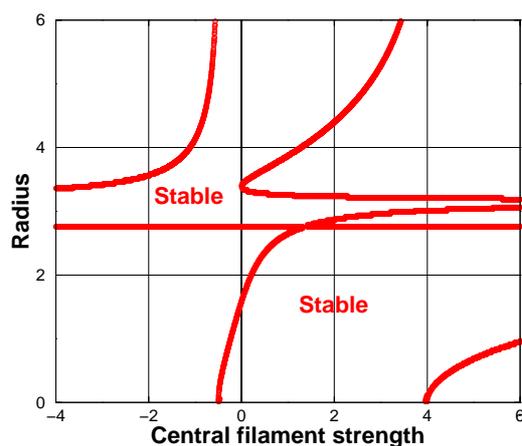


Figure 3: The stability of a ring consisting of five Alfvén filaments and a central filament of relative strength  $\kappa_0$ . The curves indicate those configurations which are neutrally stable, separating the stable and unstable regions.

## References

- [1] V. P. Lakhin, T. J. Schep, and E. Westerhof, *Phys. Plasmas* **5**(11), 3833 (1998).
- [2] J. Bergmans, B. N. Kuvshinov, V. P. Lakhin, T. J. Schep, and E. Westerhof, *Plasma Phys. Control. Fusion* **41**, A709 (1999).
- [3] J. Bergmans, B. N. Kuvshinov, V. P. Lakhin, and T. J. Schep, *Phys. Plasmas* **7**(6), 2388 (2000).
- [4] T. H. Havelock, *Phil. Mag. (7)* **11**, 617 (1931); G. J. Mertz, *Phys. Fluids* **21**, 1092 (1978).
- [5] G. K. Morikawa and E. V. Svensson, *Phys. Fluids* **14**, 1058 (1971).