

Nonlinear Landau damping in the relativistic and non-relativistic regime

M. Brunetti^{1,2}, F. Califano², F. Pegoraro^{1,2}

¹ *Dip. Fisica, Università di Pisa, Pisa, Italy*

² *Istituto Nazionale Fisica della Materia, Sez. A, Dip. Fisica, Pisa, Italy*

The self-consistent damping of longitudinal waves in collisionless plasmas is a classic fundamental problem in the study of nonlinear wave-particle interaction processes.

In the linear non-relativistic regime, when a spatially uniform plasma with equilibrium electron distribution function (e.d.f) $f^{eq}(v)$ is perturbed by a small amplitude electrostatic disturbance, the Landau's analysis [1] predicts that the time-asymptotic evolution of the electric field exhibits exponential damping (or growth) as well as oscillatory behaviour. The damping (or growing) rate is proportional to the derivative with respect to v of the equilibrium e.d.f. calculated at the phase velocity v_{ph} of the electrostatic wave. In the case of a Maxwellian equilibrium e.d.f., a long-wavelength electron plasma oscillation with $k\lambda_D \ll 1$, where λ_D is the Debye length and k is the wave number, decays with time on a scale large compared to the oscillation time. However, even in the case of 'small' amplitude perturbations, the linear analysis breaks down for $t > t_p$, t_p being the particle trapping time-scale which depends on the electric field amplitude E as $t_p = 1/\sqrt{kE}$ in normalized units (see below Eq. (2)). Thus, Landau's linear solution holds at large times only if initially the condition $t_d \ll t_p$ is satisfied, where t_d is the damping time-scale. In the opposite limit $t_d \gg t_p$, O'Neil [2] has shown that the energy exchange between the wave and the particles with velocities $v \simeq v_{ph}$ trapped in the wave prevents the complete damping of the wave which reaches a constant nonzero value asymptotically.

Recently, the existence of a critical initial perturbation amplitude, that marks the transition between these two different asymptotic regimes, has been proved in the limit of small amplitude waves [3] and the analytical expression of the asymptotic wave amplitude a_{fin} has been given in the case of a sinusoidal perturbation of a linearly stable equilibrium. In this limit, the general solution for the asymptotic electric field has been found to be a finite superposition [4] of traveling Bernstein-Greene-Kruskal (BGK) waves [5] plus higher order terms (see Eq. (9) of Ref. [3]).

We report the results of a systematic numerical study [6] of the Vlasov-Poisson equations in the intermediate range where the restrictions assumed in the papers of Landau ($t_d \ll t_p$) and O'Neil ($t_d \gg t_p$) do not apply. We solve the one-dimensional Vlasov-Poisson system of equations numerically

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = 0, \quad \frac{\partial E}{\partial x} = 1 - \int_{-\infty}^{+\infty} f dv \quad (1)$$

where $f(x, v, t)$ is the e.d.f. and $E(x, t)$ is the electric field, which at $t = 0$, is given by

$$E(x, 0) = 2 a_{in} \sin(kx) \quad (2)$$

In Eqs. (1)-(2) and in the following, time is normalized to the inverse of the electron non-relativistic plasma frequency ω_{pe} , velocity to the speed of light c and consequently

E to $mc\omega_{pe}/e$ and f to the equilibrium particle density n_0 . We assume that the ions form a fixed, neutralizing background. Oscillations are excited by initializing Fourier spatial modes with wave number k and $-k$:

$$f(x, v, 0) = f^{eq}(v)[1 + \epsilon \cos(kx)] \quad (3)$$

where ϵ is the perturbation amplitude, related to the initial electric field amplitude a_{in} as $\epsilon = 2ka_{in}$, and $f^{eq}(v)$ is the Maxwellian e.d.f., $f^{eq}(v) = (1/\sqrt{2\pi} v_{th}) \exp(-v^2/(2v_{th}^2))$ (here v_{th} is the thermal velocity). The Vlasov-Poisson equations are integrated numerically in phase space (x, v) by using the well known ‘‘splitting scheme’’ [7]. The number of points used in the simulations are typically $N_x = 256$, $N_v = 3000$ with $dt \simeq 0.0025$. The numerical phase space is given by $0 \leq x < L$, $-v_{max} \leq v \leq v_{max}$, where L is the maximum length of the space interval and v_{max} the maximum velocity which can be reached by the particles. Periodic boundary conditions are used in the space direction.

Simulations were run with $v_{max} = 0.6$, $\lambda = 2\pi/k = L$ and with different values of the initial perturbation amplitude ϵ . The thermal velocity was set to $v_{th} = 0.1$, in order to describe plasmas with a non-relativistic temperature, and the initially perturbed wave number to $k = 4$. The evolution of the system is investigated up to $t \simeq 2500 \omega_{pe}^{-1}$.

In Fig. 1 (left), we plot the asymptotic amplitude a_{fin} of the electric field versus the initial perturbation amplitude ϵ , at fixed wave number $k = 4$. The dashed line corresponds to the analytical expression of the asymptotic amplitude given by Eq. (19) in Ref. [3], which in our units and for an initial perturbation of the form given in Eq. (3), reads (for $v_{th} = 0.1$ and $v_{ph} = 0.31$)

$$a_{fin} = \frac{4\pi v_{th}^2}{(v_{ph}^2/v_{th}^2 - 1)} (\epsilon - \epsilon^*) \simeq 0.0146 (\epsilon - \epsilon^*) \quad (4)$$

where ϵ^* is the critical initial perturbation amplitude which marks the transition between the Landau’s and O’Neil’s scenarios. The analytical values a_{fin} of Eq. (4) (dashed line in Fig. 1 (left)) agree with our numerical results in the limit of small initial perturbation amplitudes $\epsilon < 0.04$ if we set $\epsilon^* \simeq 0.012$, which corresponds to $q^* = \sqrt{2/\epsilon^*}/t_d \simeq 0.85$. For initial perturbations with $\epsilon < \epsilon^*$, the field amplitude is completely damped. Thus, this method allows us to find the critical value ϵ^* avoiding the regime of very-small initial perturbations, where the simulations are particularly delicate since the electric field amplitude becomes so small that a very high numerical resolution is necessary. The critical initial value q^* differs from those previously predicted in Refs. [8], which were obtained for lower resolution and shorter times.

When the initial perturbation amplitude is such that $\epsilon > \epsilon^*$, nonlinear effects come into play and the electric field amplitude oscillates at large times around a constant nonzero value a_{fin} . After the initial linear damping of the wave, two vortices appear in phase space, centered in $v = \pm v_{ph}$, and propagate in opposite directions. In order to analyse how these results depend on the numerical resolution used in our simulations, we have performed runs with different numbers of grid points N_x . We note that the time at which the damping stops and the electric field amplitude starts to oscillate around a constant nonzero value does not depend on N_x ($t_c \simeq 950$ for $\epsilon = 0.1$). The difference between the runs is that for bigger values of N_x , larger low-frequency oscillations in the electric field are observed. In Fig. 1 (right), the contour plot of the e.d.f. in the resonant region is shown for $t = 1200$, $\epsilon = 0.1$, and for different values of the

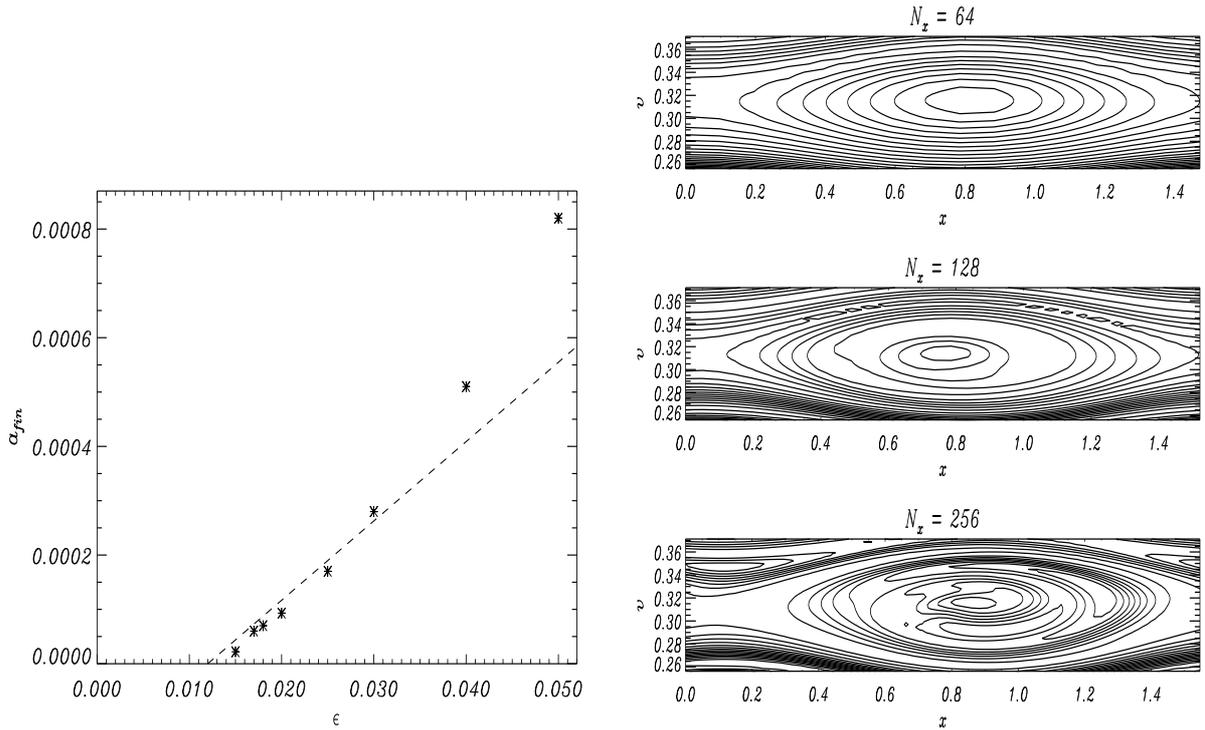


Figure 1: (Left) Final electric field amplitude a_{fim} as a function of the initial perturbation amplitude ϵ ; (Right) Contour plot of the e.d.f. in the resonant region at $t = 1200$ for $\epsilon = 0.1$, $L = \lambda$ and $N_x = 64$, 128 and 256, respectively.

spatial resolution $N_x = 64$, 128 and 256. We note that the vortices become regular, and the e.d.f. and E_k become time independent, faster as the numerical dissipation in phase space increases (i. e., for smaller N_x). Nevertheless, the overall structure which corresponds to a superposition of two traveling BGK waves is the same at large times in all the simulations since t_c , v_{ph} and the dimensions of the vortices do not change with the numerical resolution.

A similar conclusion has been obtained analytically in the small amplitude limit in Ref. [3]. In fact, in this limit the general solution for the asymptotic electric field E is a finite superposition of traveling waves plus higher order terms (see Eq. (9) of Ref. [3]). Our numerical results show that the asymptotic electric field is given by a superposition of traveling ‘averaged’ BGK waves also in the case of non-small amplitudes.

We are interested in generalizing these results to the relativistic regime [9], namely, to plasmas with temperatures $k_B T \gtrsim mc^2$ and waves such that the resonance between the particles and the wave occurs at relativistic velocities $v \simeq v_{ph} \lesssim c$.

In the relativistic regime, our mathematical model is the one-dimensional Vlasov-Poisson system of equations

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial p} = 0, \quad \frac{\partial E}{\partial x} = 1 - \int_{-\infty}^{+\infty} f dp \quad (5)$$

where $f(x, p, t)$ is the e. d. f., and all the quantities are normalized as in the non-relativistic case. Oscillations are excited by initializing Fourier spatial modes with wave

number $k = 2\pi/L$ and $-k$, as described by Eq. (3), where now $f^{eq}(p)$ is the Jüttner-Synge distribution function [10], $f^{eq}(p) = N \exp[-z \Gamma(p)]$ (here N is a normalization constant, p is the one-dimensional spatial component of the momentum, $\Gamma = \sqrt{1 + p^2}$, and z is a dimensionless parameter $z = mc^2/k_B T$).

We have studied how t_d and t_p at $t = 0$ depend on the initial temperature. The damping time t_d and the wave number k are given in the laboratory frame of reference (LAB). In order to calculate t_p as a function of quantities in LAB, we use the usual relativistic relation for the dilatation of times and write

$$t_p = \Gamma_{ph} t_{p0} = \Gamma_{ph} \sqrt{\frac{1}{k_0 a_{in0}}} = \Gamma_{ph}^{3/2} \sqrt{\frac{1}{k a_{in}}}, \quad (6)$$

where the index 0 labels the quantities in the local frame moving with $v = v_{ph}$ in LAB, and $\Gamma_{ph} = \Gamma(v_{ph})$. As we have checked from the numerical simulations performed, t_d and v_{ph} change with the temperature. Thus, t_p changes with z because of the factor $\Gamma_{ph}^{3/2}$ in eq. (6). Simulations were run differing only in the values of the initial plasma temperature, with $k = 1.6$, as the fundamental wave number, and $\epsilon = 0.1$, as the initial perturbation amplitude (in this case, for non-relativistic temperature $z \gg 1$, the initial ordering is $t_p < t_d$).

The initial time-scales ordering is found to be reversed in the regime $v_{ph} < 1$ as z decreases from 10 to 2.5. If T^* is the initial temperature at which the ordering is reversed, the long-time evolution is found to depend strongly on T^* : for $T > T^*$, the electric field amplitude damps to zero if $v_{ph} < 1$ or remains at the initial constant value if $v_{ph} > 1$ (no resonance); for $T < T^*$, the formation of vortices in phase space indicates that the electrons can gain energy from their interactions with the wave and that the electric field amplitude, after an initial damping, reaches a constant nonzero value.

References

- [1] L. D. Landau, *J. Phys.* **10**, 25 (1946).
- [2] T. O'Neil, *Phys. Fluids* **8**, 2255 (1965).
- [3] C. Lancellotti and J. J. Dornig, *Phys. Rev. Lett.* **81**, 5137 (1998).
- [4] M. Buchanan and J. J. Dornig, *Phys. Rev. Lett.* **70**, 3732 (1993).
- [5] M. Bernstein, J. M. Greene and M. D. Kruskal, *Phys. Rev.* **108**, 546 (1957).
- [6] M. Brunetti, F. Califano and F. Pegoraro, to appear in *Phys. Rev. E*.
- [7] C. Z. Cheng and G. Knorr, *J. Comp. Phys.* **22**, 330 (1976).
- [8] R. Sugihara and T. Kamimura, *J. Phys. Soc. Japan* **33**, 206 (1972); J. Canosa, *Phys. Fluids* **17**, 2030 (1974); J. J. Rasmussen, *Physica Scripta* **T2/1**, 29 (1982).
- [9] M. Brunetti, F. Califano and F. Pegoraro, *Physica Scripta* **T84**, 168 (2000).
- [10] T. L. Synge, "The Relativistic Gas" (North-Holland, Amsterdam, 1957), p. 33.
- [11] B. Buti, *Phys. Fluids* **5**, 1 (1962).