Plasma Equilibrium with Anisotropic Pressure in the Magnetic Field of a Point Dipole and its Stability

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Abstract

The interchange and ballooning stability of general anisotropic pressure plasma equilibria in a dipolar magnetic field is investigated. A fluid form of the anisotropic pressure Energy Principle is derived, which, after appropriate minimization, gives an interchange stability condition and an integro-differential ballooning equation. These results are applied to the case of an anisotropic pressure equilibrium having the perpendicular pressure equal to the parallel pressure times a constant and, in particular, to a model point dipole equilibrium. It is found that the model equilibrium is interchange stable for all plasma betas = (plasma pressure/magnetic pressure) and ballooning stable for all betas up to some critical value.

1. Introduction

Plasma stability of dipolar magnetic field equilibria is of interest for both cyclotron heated plasma laboratory experiments, such as the Levitated Dipole Experiment (LDX) [1], and space and astrophysical plasma applications [2], where the effects of anisotropic pressure should be considered.

A model point dipole plasma equilibrium with isotropic and anisotropic pressure was studied in Refs. [3] and [4] respectively, and the resulting Grad-Shafranov equation was shown to permit a relatively simple separable solution. In the isotropic plasma pressure case the equilibrium exists for arbitrarily large plasma beta = (plasma pressure/magnetic pressure) [3], while in the case of anisotropic plasma pressure a stable equilibrium is possible only for plasma betas below some critical value [4]. At high beta the equilibrium is destroyed either by the fire hose or mirror mode instability, depending on whether the parallel or perpendicular pressure is larger, respectively.

The present work studies interchange and ballooning mode stability for a general anisotropic pressure plasma equilibrium in an axisymmetric magnetic field with closed field lines and then applies these results to the case of the model point dipole equilibrium of Ref. [4].

2. General Stability Properties of Anisotropic Plasma Pressure Equilibrium

For an axisymmetric dipolar magnetic configuration with only a poloidal magnetic field all equilibrium plasma currents are in the toroidal direction so that there is no parallel current flow along the magnetic field. All magnetic field lines are closed so that "flux" or pressure surfaces are defined by their surfaces of rotation about the symmetry axis of the system.

To investigate the stability of an anisotropic pressure plasma equilibrium in a dipolar magnetic field the Kruskal-Oberman form of the Energy Principle [5, 6] can be used with the parallel current density term set to zero. The stabilizing plasma compressibility term is derived using a kinetic theory approach and is given in terms of integrals along particle trajectories, which can be bounded by a fluid form using the Schwarz inequality [6]. The expression for the potential energy $W \propto \omega^2$ (where ω is the mode frequency) from Ref. [6] can be rewritten in the form (see Ref. [7] for details)

$$W = W_{\mathrm{fluid}} + W_{\mathrm{kinetic}}, \tag{1}$$

with

$$\begin{aligned} W_{\mathrm{fluid}} \equiv \int d^3r \Bigg\{ & \frac{Q_\perp^2 (1-\sigma_-)}{8\pi} + \frac{B^2(1+\sigma_\perp)}{8\pi}\left[\nabla\cdot\vec{\xi}_\perp + \left(\vec{\kappa}\cdot\vec{\xi}_\perp\right)\Sigma_-\right]^2 \\ & + \frac{B^2(1-\sigma_-)}{8\pi}\Sigma_+\left(\vec{\kappa}\cdot\vec{\xi}_\perp\right)^2 - \frac{1}{2}\left(\vec{\kappa}\cdot\vec{\xi}_\perp\right)\left(\nabla\left(p_\parallel + p_\perp\right)\cdot\vec{\xi}_\perp\right)\Bigg\}, \end{aligned} \tag{2}$$

$$W_{\mathrm{kinetic}} \geq \frac{1}{2}\int d\zeta \int d\psi \left\{ \frac{\left[\oint (d\ell/B)\left[\left(\nabla\cdot\vec{\xi}_\perp\right)\Gamma_1 - \left(\vec{\kappa}\cdot\vec{\xi}_\perp\right)\Gamma_2\right]\right]^2}{\oint (d\ell/B)\Gamma_3} \right\}, \tag{3}$$

where $\vec{Q} = \nabla\times\left(\vec{\xi}_\perp \times \vec{B}\right)$ with $\vec{\xi}_\perp$ the perpendicular plasma displacement; $\vec{B} = \nabla\psi\times\nabla\zeta$ is the dipole magnetic field, with ζ the toroidal angle variable and ψ the poloidal flux function; $p_\parallel$, $p_\perp$ and B are the parallel and perpendicular plasma pressure and magnetic field magnitude; $\vec{\kappa}$ is the magnetic field line curvature; $\Sigma_+ = (\sigma_- + \sigma_\perp)/(1+\sigma_\perp)$ and $\Sigma_- = 1 + (1-\sigma_-)/(1+\sigma_\perp)$ with $\sigma_- = 4\pi\left(p_\parallel - p_\perp\right)/B^2$, $\sigma_\perp = 4\pi\,(C + 2p_\perp)/B^2$ and $C = \int d^3v\left\{(\mu B)^2(\partial F/\partial\varepsilon)\right\}$; $\Gamma_1 = p_\perp/2 - C$, $\Gamma_2 = p_\perp/2 + 3p_\parallel/2 + C$ and $\Gamma_3 = p_\perp + 3p_\parallel/4 - C$; and $d\ell$ is an incremental distance along the dipole magnetic field.

We begin by noting that the higher the toroidal mode number n, the more unstable the mode (see Ref. [8] for details), and then minimize W with respect to the toroidal component of the plasma displacement. Then we are left with

$$\vec{\xi}_\perp = \left(\xi/R^2B^2\right)\nabla\psi, \qquad Q_\perp^2 = R^{-2}B^{-2}\left(\vec{B}\cdot\nabla\xi\right)^2$$

and perform the minimization with respect to ξ of the functional

$$\Lambda = \frac{W}{H} \propto \omega^2,$$

where $H = (1/2)\int d^3r\,\rho\,\xi_\perp^2$ and ρ is the mass density. Considering interchange modes ($\vec{B}\cdot\nabla\xi = 0$) gives the general finite beta interchange stability condition [7]

$$\oint\frac{d\ell}{B}\left\{\frac{1}{B^2R^2}\left[\Sigma_+\,\vec{\kappa}\cdot\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) - \vec{\kappa}\cdot\nabla\left(p_\parallel + p_\perp\right)\right]\right\} + \frac{\left\{\oint\frac{d\ell}{B}\,\frac{\vec{\kappa}\cdot\nabla\psi}{B^2R^2}\,\left[\Sigma_-\,\Gamma_1 + \Gamma_2\right]\right\}^2}{\oint\frac{d\ell}{B}\left\{\Gamma_3 + \frac{4\pi\Gamma_1^2}{B^2(1+\sigma_\perp)}\right\}} \geq 0, \tag{4}$$

while the general, infinite n, Euler integro-differential ballooning equation ($Q_\perp^2 \neq 0$) can be shown to be [7]

$$\begin{aligned} & R^2B^2\,\vec{B}\cdot\nabla\left(\frac{1-\sigma_-}{R^2B^2}\,\vec{B}\cdot\nabla\xi\right) + 4\pi\left[\vec{\kappa}\cdot\nabla\left(p_\parallel + p_\perp\right) - \Sigma_+\,\vec{\kappa}\cdot\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) + \rho\Lambda\right]\xi \\ & = 4\pi(\vec{\kappa}\cdot\nabla\psi)\,\left[\Sigma_-\,\Gamma_1 + \Gamma_2\right]\,\frac{\oint\frac{d\ell}{B}\left\{\frac{\vec{\kappa}\cdot\nabla\psi}{B^2R^2}\,\xi\,\left[\Sigma_-\,\Gamma_1 + \Gamma_2\right]\right\}}{\oint\frac{d\ell}{B}\left\{\Gamma_3 + \frac{4\pi\Gamma_1^2}{B^2(1+\sigma_\perp)}\right\}}. \end{aligned} \tag{5}$$

As the physical system is symmetric with respect to the equatorial plane, Eq. (5) has two families of solutions: (1) up-down anti-symmetric, or "odd", with $\xi = 0$ at the equatorial plane; and (2) up-down symmetric, or "even", with $\vec{B}\cdot\nabla\xi = 0$ at the equatorial plane.

As in Ref. [8], some important properties of the eigenvalues Λ_j of Eq. (5) can be determined by considering the eigenvalues λ_j of the corresponding homogeneous differential equation - Eq. (5) without the right-hand side. Here j = 0, 1, 2, …, with odd (even) j's corresponding to the odd (even) modes. In particular it can be shown [7] that $\Lambda_{2j+1} = \lambda_{2j+1}$ and $\lambda_{2j} \le \Lambda_{2j} \le \lambda_{2j+2} \le \Lambda_{2j+2}$. Consequently, if $\lambda_0 \ge 0$ and $\lambda_1 \ge 0$ then $\Lambda_j \ge 0$ and the equilibrium is stable; while if $\lambda_{2j+2} < 0$ then $\Lambda_{2j} < 0$ and if $\lambda_{2j+1} < 0$ then $\Lambda_{2j+1} = \lambda_{2j+1} < 0$, so that the low j ballooning eigenmodes are unstable. In the more subtle case when $\lambda_0 < 0 < \lambda_2$ it is not clear if Λ_0 is positive (stable), negative (unstable) or changes sign. However, this question can be resolved when the parallel and perpendicular plasma pressures are proportional to each other: $p_\perp = (1+2p)p_\parallel$, with p = constant an anisotropy parameter. In particular it can be shown [7] that the lowest even mode of Eq. (5) is ballooning stable if it is interchange stable and vice versa. Therefore, we are able to conclude that the equilibrium with $p_\perp = (1+2p)p_\parallel$ is ballooning stable if $\Lambda_1 = \lambda_1 > 0$, $\lambda_2 > 0$ and the equilibrium is interchange stable.

3. Stability of Anisotropic Plasma Pressure Equilibrium for a Point Dipole

Next, we consider the case of the point dipole equilibrium of Ref. [4] as a specific example of an equilibrium with $p_\perp = (1+2p)p_\parallel$. Stable anisotropic pressure plasma equilibria do not exist for all plasma betas. For $p_\parallel > p_\perp$ there is the fire-hose beta limit β_{fh} (given by $1-\sigma_- = 0$), and for $p_\perp > p_\parallel$ there is a mirror mode stability beta limit β_{mm} (given by $1+\sigma_\perp = 0$). For the case of the plasma equilibrium under consideration these two conditions can be written as constraints on the equilibrium equatorial plane plasma beta, β_0, and anisotropy parameter, p (p > -1/2), as (see Ref. [4])

$$\beta_{mm} \equiv \frac{1+p}{p(1+2p)} > \beta_0 \quad \text{and} \quad \beta_{fh} \equiv -\frac{1+p}{p} > \beta_0, \tag{6}$$

where $\beta \equiv 4\pi(p_\parallel + p_\perp)/B^2$ equals the constant β_0 everywhere in the equatorial plane [4].

Applying inequality (4) to the equilibrium of Ref. [4] it can be shown that this equilibrium is always interchange stable. Then, the results of Sec. 2 can be applied to determine the ballooning stability of the equilibrium. The ballooning equation (5) (or its homogeneous analog) must be solved numerically. The numerical solutions show [7] that the behavior of the eigenvalues clearly reflects the presence of the fire-hose and mirror mode beta limits at the values given by Eq. (6). The first even eigenmode of the homogeneous differential form of Eq. (5) (without the right hand side) is unstable (i.e. $\lambda_0 < 0$) for any beta, while the first odd eigenmode and the second even eigenmode are stable (i.e. $\lambda_1, \lambda_2 > 0$) up to some critical beta - below or at the corresponding fire-hose or mirror mode beta limit. The eigenvalues of the integro-differential form of Eq. (5) (i.e. with the right hand side), Λ_j, are bigger than or equal to the corresponding eigenvalues of the homogeneous equation, λ_j, for all betas, because of the stabilizing influence of plasma compressibility. In particular, the lowest even mode of the integro-differential equation is stable (i.e. $\Lambda_0 > 0$) for all betas up to a critical value. For the equilibria of Ref. [4] and for $p_\perp/p_\parallel > 1$, the beta limit is set by the mirror instability if $p_\perp/p_\parallel > 8$ (i.e. p > 7/2) and by the ballooning mode if $p_\perp/p_\parallel < 8$ (i.e. p < 7/2).

4. Conclusions

The stability of a general anisotropic plasma pressure equilibrium in a dipole magnetic field was investigated. The Kruskal-Oberman form of the Energy Principle was rewritten, using a Schwarz inequality, to give a “fluid” form of MHD Energy Principle. This expression was minimized with respect to the component of plasma displacement in the toroidal direction to give a finite beta interchange stability condition and an integro-differential ballooning equation.

The general stability theory was then applied to a plasma equilibrium with $p_\perp = (1+2p)p_\parallel$. It was found that for the periodic plasma displacements, the plasma equilibrium is ballooning stable if it is interchange stable and the first odd and second even modes of the “simplified ballooning equation” are stable. Unlike the tokamak case where there are regions of favorable and unfavorable magnetic field line curvature, magnetic field line curvature of the dipole field is always unfavorable. On the other hand, unlike the tokamak case, magnetic field lines are closed for a dipole equilibrium, which provides a stabilizing plasma compressibility term (for the even modes) in addition to the usual magnetic field line bending stabilization.

Finally, the stability of the separable point dipole plasma equilibrium of Ref. [4] was investigated. Both the full integro-differential ballooning equation and the simplified differential equation were solved numerically. It was found that for periodic boundary conditions the equilibrium is interchange stable for all plasma betas and ballooning stable for all betas up to some critical value, which is below the mirror mode beta limit β_{mm} when the pressure anisotropy parameter $p < 7/2$ (i.e. $p_\perp/p_\parallel < 8$) and is equal to β_{mm} for $p > 7/2$ (i.e. $p_\perp/p_\parallel > 8$).

The field line tied boundary conditions case is more stable than the periodic boundary conditions case because of the additional line bending stabilization in the even modes.

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