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DISPERSION EQUATION OF THE CYCLOTRON WAVES IN A TWO-DIMENSIONAL MAGNETOSPHERIC PLASMA WITH ANISOTROPIC PARTICLE ENERGY

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The energetic particles (electrons, protons, heavy ions) with anisotropic temperature can lead to a wide class of ion/electron cyclotron wave instabilities in the Earth's magnetosphere, see, e.g., Ref. [1] and bibliography therein. These instabilities should be analyzed by solving the Vlasov-Maxwell's equations, taking into account a two-dimensional nonuniformity of the geomagnetic field and bounce-resonant wave-particle interactions there, Refs. [2-4]. In this paper, the linearized Vlasov equation is solved for trapped particles with the bi-maxwellian steady-state distribution function in an axisymmetric magnetospheric plasma with circular magnetic field lines: $B(R,\phi) = B_0R_0^3/(R^3\cos\phi)$. Here, R_0 is the radius of the Earth, R is the geocentric distance, ϕ is the geomagnetic latitude, B_0 is the magnetic field in an equatorial plane on the Earth's surface (at the point where $R = R_0, \phi = 0$). To solve the Vlasov equation we use the standard method of switching to new variables associated with the conservation integrals of energy: $v_\perp^2 + v_\parallel^2 = const$, magnetic moment: $v_\perp^2/2B = const$, and the equation of the $\mathbf{B}$-field line: $R/\cos\phi = const$. Introducing the variables $v = \sqrt{v_\parallel^2 + v_\perp^2}$, $\mu = v_\perp^2 B(L,0)/(v^2B(L,\phi))$, $L = R/(R_0\cos\phi)$ (instead of $v_\parallel, v_\perp, R$) we seek the perturbed distribution function as

$$f(t,R,\phi,\theta,v_\parallel,v_\perp,\alpha) = \sum_s^{\pm1}\sum_l^{\pm\infty} f_l^s(\phi,L,v,\mu)\exp(-i\omega t + im\theta + il\alpha),$$

where α is the gyrophase angle in velocity space. So that the linearized Vlasov equation for harmonics f_0^s and $f_{\pm1}^s$ can be rewritten in the next form:

$$\sqrt{1-\frac{\mu}{\cos^4\phi}}\frac{\partial f_l^s}{\partial\phi} - is\frac{LR_0}{v}\left(\omega + \frac{l\,\omega_{co}}{L^3\cos^4\phi}\right)f_l^s = Q_l^s, \qquad l = 0, \pm1, \tag{1}$$

where
$$Q_0^s = \frac{e}{T_\parallel}R_0L\sqrt{1-\frac{\mu}{\cos^4\phi}}F_0E_\parallel, \qquad E_{\pm1} = E_n \mp iE_b,$$

$$Q_{\pm1}^s = \frac{e}{2T_\perp}R_0L\frac{\sqrt{\mu}}{\cos^2\phi}F_0\left[sE_{\pm1} - i\frac{v\cos^2\phi}{\omega R_0L}\left(\frac{T_\perp}{T_\parallel}-1\right)\sqrt{1-\frac{\mu}{\cos^4\phi}}\frac{\partial}{\partial\phi}\frac{E_{\pm1}}{\cos^2\phi}\right],$$

$$F_0 = \frac{N(L)}{\pi^{1.5}v_{T\parallel}v_{T\perp}^2}\exp\left\{-\frac{v^2}{v_{T\parallel}^2}\left[1-\frac{\mu}{\cos^4\phi}\left(1-\frac{T_\parallel}{T_\perp}\right)\right]\right\}, \quad v_{T\parallel}^2 = \frac{2T_\parallel}{M}, \quad v_{T\perp}^2 = \frac{2T_\perp}{M}.$$

Here, $E_\parallel, E_n, E_b$ are, respectively, the parallel, normal and binormal perturbed electric field components relative to $\mathbf{B}$; F_0 is the steady-state distribution function of plasma particles with the density N, parallel and perpendicular temperature $T_\parallel$ and $T_\perp$, respectively, charge e and mass M. By the indexes $s = \pm1$ we differ the particles with positive and negative values of $v_\parallel = sv\sqrt{1-\mu/\cos^4\phi}$ relatively to $\mathbf{B}$. In Eq. (1) we have neglected the drift corrections assuming the wave frequency ω is much larger than the drift frequency, that is valid when $mv_{T\perp}^2L^2/(v_{T\parallel}R_0\omega_{co}) << 1$, where $\omega_{co} = eB_0/Mc$, and m is the azimuthal wave number over θ (east-west) direction. Depending on μ, the domain of perturbed distribution functions is defined by the inequalities $L^{-4} \le \mu \le 1$ and $-\phi_t(\mu) \le \phi \le \phi_t(\mu)$, where $\pm\phi_t(\mu)$ are the local mirror points for the trapped particles at a given (by L) magnetic field line, which are defined by the zeros of parallel velocity. As a result, $\phi_t = \arccos\ \mu^{0.25}$.

After solving Eq. (1), the two-dimensional longitudinal (parallel to $\mathbf{B}$), $j_{\|}(\phi, L)$, and transverse, $j_{\pm 1}(\phi, L)$, current density components can be expressed as

$$j_{\|}(\phi, L) = \frac{\pi e}{\cos^4\phi} \sum_s^{\pm 1} s \int_0^{\infty} v^3 \int_{L^{-4}}^{\cos^4\phi} f_0^s(\phi, L, v, \mu)\, d\mu\, dv, \tag{2}$$

$$j_l(\phi, L) = \frac{\pi e}{2\cos^4\phi} \sum_s^{\pm 1} \int_0^{\infty} v^3 \int_{L^{-4}}^{\cos^4\phi} \frac{\sqrt{\mu} f_l^s(\phi, L, v, \mu)}{\sqrt{\cos^4\phi - \mu}} d\mu\, dv, \qquad l = \pm 1. \tag{3}$$

Note, the normal and binormal to $\mathbf{B}$ current density components in our notation are equal to $j_n = j_1 + j_{-1}$ and $j_b = i(j_1 - j_{-1})$, respectively. In this paper, we evaluate the transverse dielectric permittivity. The longitudinal permittivity elements are derived by analogy in Ref. [5]. Note, the longitudinal permittivity in magnetospheric plasmas with an equilibrium distribution function (when $T_{\|} = T_{\perp}$) has been evaluated in Ref. [6] for two plasma models with dipole and circular magnetic field lines. Accounting that the trapped particles, with a given μ, execute the periodic motion with the bounce period proportional to

$$\tau_b = \tau_b(\mu) = 4 \int_0^{\phi_t} \frac{\cos^2\phi}{\sqrt{\cos^4\phi - \mu}} d\phi\ ,$$

the solution of Eq. (1) (for harmonics with $l = \pm 1$) is

$$f_l^s(\phi, L, v, \mu) = \sum_{p=-\infty}^{+\infty} f_{l,p}^s(L, v, \mu) \exp\left[ip\frac{2\pi}{\tau_b}\tau(\phi) + isl\frac{R_0\omega_{co}}{L^2 v}C(\phi)\right], \tag{4}$$

where

$$\tau(\phi) = \int_0^{\phi} \frac{\cos^2\eta\, d\eta}{\sqrt{\cos^4\eta - \mu(\kappa)}}, \quad C(\phi) = \int_0^{\phi} \frac{d\eta}{\cos^2\eta\sqrt{\cos^4\eta - \mu(\kappa)}} - \frac{2E(\kappa) - K(\kappa)}{(1-2\kappa)^2\Pi(\kappa)}\tau(\phi).$$

The perturbed distribution functions, defined by Eq. (4), satisfy automatically the corresponding boundary conditions for the trapped particles, namely, the continuity of the distribution functions ($f_l^{+1} = f_l^{-1}$) at the reflection points $\pm\phi_t$. Here, a new variable $\kappa = 0.5(1 - \sqrt{\mu})$ is introduced instead of μ-variable: $\mu(\kappa) = (1-2\kappa)^2$. So that

$$\phi_t = \arcsin\sqrt{2\kappa}, \qquad \tau_b(\kappa) = 2\sqrt{2(1-2\kappa)}\Pi(\kappa), \qquad E(\kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa\sin^2\phi}\, d\phi,$$

$$\Pi(\kappa) = \int_0^{\pi/2} \frac{d\phi}{(1 - 2\kappa\sin^2\phi)\sqrt{1-\kappa\sin^2\phi}}, \qquad K(\kappa) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\kappa\sin^2\phi}},$$

where $K(\kappa)$, $E(\kappa)$ and $\Pi(\kappa)$ are the complete elliptic integrals of the first, second, and third kind, respectively.

After the s-summation and using the Fourier expansion of $E_{\pm 1}$ over the ϕ-angle, see Eq. (7), the transverse current density component can be expressed as

$$\frac{4\pi i}{\omega} j_l(L, \phi) = \frac{\omega_{po}^2 R_0 L v_{T\|}}{\omega \pi^{1.5} v_{T\perp}^2 \cos^4\phi} \sum_{n'=-\infty}^{+\infty} E_l^{(n')} \sum_{p=-\infty}^{+\infty} \int_{0.5\sin^2\phi}^{(L^2-1)/2L^2} \frac{(1-2\kappa)^3 d\kappa}{\sqrt{\cos^4\phi - \mu(\kappa)}} \times$$

$$\times \int_{-\infty}^{+\infty} \frac{u^4 \exp(-u^2)}{pu - Z_l(\kappa)} \exp\left(ip\frac{2\pi}{\tau_b}\tau(\phi) + il\frac{R_0\omega_{co}}{L^2 u v_{T\|}}C(\phi)\right) A_{p,l}^{n'}(u, \kappa) du, \tag{5}$$

where

$$\omega_{po}^2 = \frac{4\pi N e^2}{M}, \qquad u = \frac{v}{v_{T\parallel}}, \qquad Z_l(\kappa) = \frac{R_0 L \tau_b}{2\pi v_{T\parallel}} \left[\omega + l \frac{\omega_{co}}{L^3} \frac{2E(\kappa) - K(\kappa)}{(1-2\kappa)^2 \Pi(\kappa)} \right],$$

$$A^n_{p,l}(u,\kappa) = \int_{-\phi_t}^{\phi_t} \exp\left[\frac{u^2\mu(\kappa)}{\cos^4\phi}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right] \Phi^n_{p,l}(u,\kappa,\phi) \frac{\cos^2\phi \, d\phi}{\sqrt{\cos^4\phi - \mu(\kappa)}} +$$

$$+(-1)^p \int_{-\phi_t}^{\phi_t} \exp\left[\frac{u^2\mu(\kappa)}{\cos^4\phi}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right] \Phi^n_{-p,l}(-u,\kappa,\phi) \frac{\cos^2\phi \, d\phi}{\sqrt{\cos^4\phi - \mu(\kappa)}} +$$

$$+\frac{\pi n u v_{T\parallel}}{\phi_o \omega R_0 L}\left(\frac{T_\perp}{T_\parallel} - 1\right)\left\{\int_{-\phi_t}^{\phi_t} \exp\left[\frac{u^2\mu(\kappa)}{\cos^4\phi}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right] \Phi^n_{p,l}(u,\kappa,\phi) d\phi + \right.$$

$$\left. + (-1)^p \int_{-\phi_t}^{\phi_t} \exp\left[\frac{u^2\mu(\kappa)}{\cos^4\phi}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right] \Phi^n_{-p,l}(-u,\kappa,\phi) d\phi \right\},$$

$$\Phi^n_{p,l}(u,\kappa,\phi) = \cos\left(\frac{\pi n}{\phi_o}\phi - p\frac{2\pi}{\tau_b}\tau(\phi) - \frac{l R_0 \omega_{co}}{L^2 u v_{T\parallel}} C(\phi)\right).$$

The points $\pm\phi_0(L) = \pm\arccos(1/L)$, in Eqs. (5,7), are the beginning and the end of a given (by L) magnetic field line on the Earth's surface. As a result, the transverse current density component in an axisymmetric magnetosphere is derived by the p-summation of the bounce-resonant terms. It should be noted that the bounce-resonance conditions, $pu - Z_l(\kappa) = 0$ in Eq. (5), for the trapped particles in magnetospheric plasmas are

$$\omega + l\frac{\omega_{co}[2E(\kappa) - K(\kappa)]}{L^3\,(1-2\kappa)^2\Pi(\kappa)} = \frac{p\,\pi\,v}{R_0 L\sqrt{2(1-2\kappa)\Pi(\kappa)}}, \qquad l = 0, \pm 1. \tag{6}$$

These conditions are entirely different from the corresponding expressions in the straight magnetic field case. Of course, as in the straight magnetic field, $l = 1$ corresponds to the effective wave-electron interaction, and $l = -1$ corresponds to the wave-ion interaction. Moreover, there is no possibility to carry out the Landau integration over the particle energy $u = v/v_{T\parallel}$ (by introducing the plasma dispersion function) because the phase coefficients $A^{n'}_{p,l}$ depend on u.

To solve the two-dimensional wave equations, we should expand preliminary the perturbed values in a Fourier series over ϕ. In particular, for the transverse components of the current density, j_l, and electric field, E_l, we have:

$$\frac{j_l(L,\phi)}{\cos^2\phi} = \sum_n^{\pm\infty} j_l{}^{(n)}(L)\ \exp\left[\frac{i\pi n\phi}{\phi_0(L)}\right], \qquad \frac{E_l(L,\phi)}{\cos^2\phi} = \sum_{n'}^{\pm\infty} E_l{}^{(n')}(L)\ \exp\left[\frac{i\pi n'\phi}{\phi_0(L)}\right]. \tag{7}$$

This procedure converts the operator, representing the dielectric tensor, into a matrix whose elements are calculated independently on the solutions of Maxwell's equations. As a result,

$$\frac{4\pi i}{\omega} j_l^{(n)}(L) = \sum_{n'}^{\pm\infty} \epsilon_l^{n,n'}(L) E_l^{(n')}(L),$$

and the contribution of a given kind of plasma particles to the transverse permittivity elements, $\epsilon_l^{n,n'}(L)$, is

$$\epsilon_l^{n,n'}(L) = \frac{\omega_{po}^2 L R_0 v_{T\parallel}^3}{\omega 2\pi^{1.5} v_{T\perp}^4 \phi_o} \sum_p^{\pm\infty} \int_0^{\frac{L^2-1}{2L^2}} (1-2\kappa)^3 d\kappa \int_{-\infty}^{+\infty} \frac{u^4 \exp(-u^2)}{pu - Z_l(\kappa)} D^n_{p,l}(u,\kappa) A^{n'}_{p,l}(u,\kappa) du, \tag{8}$$

where

$$D^n_{p,l}(u,\kappa) = \int_{-\phi_t}^{\phi_t} \cos\left(\frac{\pi n}{\phi_o}\phi - p\frac{2\pi}{\tau_b}\tau(\phi) - \frac{lR_0\omega_{co}}{L^2 u v_{T\parallel}}C(\phi)\right)\frac{d\phi}{\cos^6\phi\sqrt{\cos^4\phi - \mu(\kappa)}}.$$

Thus, due to a geomagnetic field inhomogeneity, the whole spectrum of the electric field (by $\sum_{n'}^{\pm\infty}$) is present in a given (by n) current density harmonic.

As was noted above, Eq. (8) describes the contribution of any kind of trapped particles to the transverse permittivity. The corresponding expressions for plasma electrons and ions can be obtained from (8) by replacing T (temperature), N (density), M (mass), e (charge) by the electron $T_{\parallel e}$, $T_{\perp e}$, N_e, m_e, e_e and ion $T_{\parallel i}$, $T_{\perp i}$, N_i, M_i, e_i parameters, respectively. To obtain a total expression of the transverse permittivity, as usual, it is necessary to carry out the summation over all kinds of plasma particles. The same comments should be addressed to derive a total two-dimensional transverse current, by Eq. (5).

Since the cyclotron wave instabilities are an important contributor to cool plasma heating, it is possible (and interesting) to develop a two-dimensional numerical code to describe these processes in the Earth's magnetosphere with new dielectric tensor components, taking into account the bounce-resonant effects. To have some analogy with the linear theory [1] of cyclotron wave instabilities in the straight magnetic field, let's assume that the nth harmonic of the electric field gives the main contribution to the nth harmonic of the current density (*one-mode approximation*). In this case, for the field-aligned electromagnetic cyclotron waves (when $m = 0, \partial/\partial L = 0$), there is following dispersion equation:

$$\left(\frac{\pi\ n\ c}{R_0 L\phi_o\omega}\right)^2 = 1 + 2\sum_\sigma \epsilon^{n,n}_{l,\sigma}(L) \qquad (9)$$

where σ denotes the particle spaces (electron, proton, etc.) This equation is suitable to analyze the instability of the electron-cyclotron waves if $l = 1$, and ion-cyclotron waves if $l = -1$. Note, in our notation, the parallel wave vector is defined as $k_\parallel = \pi n/(R_0 L\phi_o)$, so that $\pi nc/(R_0 L\phi_o\omega)$ is the nondimensional parallel refractive index in Eqs. (5,8,9). Further, Eq. (9) should be resolved numerically for the real and imaginary parts of the wave frequency, $\omega = \mathrm{Re}\ \omega + i\mathrm{Im}\ \omega$, to define the conditions of the ion/electron cyclotron instabilities in the magnetospheric plasma with anisotropic temperature.

Of course, the similar approach/consideration is possible as well for a magnetospheric plasma with dipole magnetic field lines.

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