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Magnetic Dipole Equilibrium of a Gravitating Plasma with Incompressible Flows

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1. Introduction and conclusions.

In a recent investigation [1] it was found that the ideal magnetohydrodynamic equilibrium of an axisymmetric gravitating magnetically confined plasma with incompressible flows is governed by a second-order elliptic differential equation for the poloidal magnetic flux function ψ containing five surface quantities coupled with a Poisson equation for the gravitation potential Ω, and an algebraic relation for the pressure. For vanishing flows this equation reduces to the Grad-Schlüter-Shafranov equation. Analytic solutions of the nonlinear Grad-Schlüter-Shafranov equation for a plasma with vanishing poloidal current at either low or high pressure confined by a dipolar magnetic field were obtained recently in Ref. [2]. These studies were then extended to equilibria with purely toroidal flow [3] and to gravitating magnetic dipolar plasmas without flow [4].

The purpose of the present work is to derive analytic magnetic dipolar equilibria for a plasma at finite pressure and poloidal current with incompressible sheared flows having non-vanishing toroidal and poloidal components, under the exertion of gravitational forces from a massive body (a star or a black hole). In Section 2 the equilibrium equations for the system under consideration are derived. Analytic magnetic dipolar solutions are constructed in Sections 3 and 4 in the following regimes: (a) in the low-energy regime $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \ll 1$, where β_0, γ, δ_0, and ϵ_0 (defined in Section 2) are related to the thermal, poloidal-current, flow, and gravitating energies normalized to the poloidal-magnetic-field energy, respectively, and (b) in the high-energy regime $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1$. It turns out that in the high-energy regime all four forces, pressure-gradient, toroidal-magnetic-field, inertial, and gravitating contribute equally to the formation of magnetic surfaces very extended and localized about the symmetry plane such that the resulting equilibria resemble the accretion disks in astrophysics.

2. Equilibrium equations

With use of cylindrical coordinates (R, ϕ, z) the equilibrium of an axisymmetric gravitating plasma with incompressible flow satisfies (in convenient units) the differential equation for ψ derived in Ref. [1] (where also other details are given)

$$(1-M^2)\Delta^\star\psi - \frac{1}{2}(M^2)'|\nabla\psi|^2 + \frac{1}{2}\left(\frac{X^2}{1-M^2}\right)' + R^2\left(P_s' - \Omega\rho'\right) + \frac{R^4}{2}\left(\frac{\rho(\Phi')^2}{1-M^2}\right)' = 0 \quad (1)$$

(stemming from the "radial" component of the force-balance equation $\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{j}\times\mathbf{B} - \nabla P - \rho\nabla\Omega$) together with $\nabla^2\Omega = G_0\rho_t$. Here, $P_s(\psi)$, $\rho(\psi)$, and $\Phi(\psi)$ are, respectively, the static pressure, density, and electrostatic potential which remain constant on magnetic surfaces $\psi =$ const.; the flux functions $F(\psi)$ and $X(\psi)$ are related to the poloidal flow and the toroidal magnetic field, respectively; $M^2 = (F')^2/\rho$ is the Mach-number of the poloidal velocity with respect to the poloidal-magnetic-field Alfvén velocity; ρ_t is the total density including contributions from the plasma itself and from external mass sources; $4\pi G_0$ is the constant of gravity; $\Delta^\star \equiv R^2\nabla\cdot(\nabla/R^2)$; and the prime denotes differentiation with respect to ψ. Because of axisymmetry the equilibrium quantities do not depend on ϕ. The magnetic field, current density, and plasma velocity can, respectively, be expressed as $\mathbf{B} = I(R,z)\nabla\phi + \nabla\phi\times\nabla\psi$, $\mathbf{j} = \Delta^\star\psi\nabla\phi - \nabla\phi\times\nabla I$, and $\rho\mathbf{v} = \Theta(R,z)\nabla\phi + \nabla\phi\times\nabla F(\psi)$. Unlike to the case of static equilibria and vanishing gravity the pressure $P = P_s(\psi) + \rho\left[v^2/2 + \Omega + \left(R^2(\Phi')^2\right)/(1-M^2)\right]$ is not a flux function.

Under the transformation $U(\psi) = \int_0^\psi [1 - M^2(g)]^{1/2}\, dg, \quad M^2 < 1$, Eq. (1) reduces (after dividing by $(1-M^2)^{1/2}$) to

$$\Delta^\star U + \frac{1}{2}\frac{d}{dU}\left(\frac{X^2}{1-M^2}\right) + R^2\left(\frac{dP_s}{dU} - \Omega\frac{d\rho}{dU}\right) + \frac{R^4}{2}\frac{d}{dU}\left[\rho\left(\frac{d\Phi}{dU}\right)^2\right] = 0. \tag{2}$$

Eq. (2) is free of the nonlinear term $1/2(M^2)'|\nabla\psi|^2$ and, therefore, for $M^2 < 1$ the equilibrium can be determined from the more tractable set of Eqs. $\nabla^2\Omega = G_0\rho_t$, (2), and the above noted relation for the pressure.

The equilibrium of a plasma confined by the magnetic field of a current ring lying on the symmetry plane and centered at the origin of the system of coordinates is now considered. Employing spherical coordinates r, θ and ϕ with $\mu = \cos\theta$ and $R = r\sin\theta$, we seek separable solutions of Eq. (2) of the form

$$U(r,\mu) = U_0 H(\mu)\left(\frac{r_0}{r}\right)^\alpha. \tag{3}$$

Here, H is an unknown function of μ alone such that $H(0) = 1$, and U_0 and r_0 are normalization constants specifying a reference flux surface location. The parameter α plays the role of an eigenvalue of equations $\nabla^2\Omega = G_0\rho_t$ and (2). It equals unity or -2 in the vacuum limit to recover the dipolar solutions $\psi_{vac} \propto (1-\mu^2)/r$ and $\psi_{vac} \propto (1-\mu^2)r^2$ describing, respectively, the flux surfaces far away from and close to the origin. We are interested in configurations symmetric with respect to the symmetry plane. Accordingly, the boundary conditions $H(\mu\to 1) \propto 1-\mu$ and $dH/d\mu|_{\mu=0} = 0$ are chosen to keep the magnetic field finite at $\theta = 0$ and parallel to the axis of symmetry at $\theta = \pi/2$, respectively. A plasma subject only to gravitating forces from a star or black hole of mass M_s placed at $r = 0$ is further considered; the plasma self gravity is neglected. Consequently, the Poisson equation for Ω decouples from (2) and has the solution $\Omega = -G_0M_s/(4\pi r)$. Inspection of Eq. (2) with this gravitation potential implies that the separable solution (3) is only possible provided $P_s = P_{s0}\left(U/U_0\right)^{2+4/\alpha}$, $X^2/(1-M^2) = X_0^2\left(U/U_0\right)^{2+2/\alpha}$, $\rho = \rho_0\left(U/U_0\right)^{2+3/\alpha}$, and $\rho\left(d\Phi/dU\right)^2 = \rho_0\left(\Phi_0/U_0\right)^2\left(U/U_0\right)^{2+6/\alpha}$, where P_{s0}, X_0, ρ_0, and Φ_0 are normalization constants associated with the reference flux surface U_0. Inserting this ansatz, Eq. (3) and $\Omega = -G_0 M_s/(4\pi r)$ into Eq. (2) we obtain

$$\frac{d^2 H}{d\mu^2} + \frac{\alpha(\alpha+1)}{1-\mu^2}H = -\beta_0\alpha(2+\alpha)H^{1+4/\alpha} - \gamma_0\alpha(1+\alpha)(1-\mu^2)^{-1}H^{1+2/\alpha} -\delta_0\alpha(3+\alpha)(1-\mu^2)H^{1+6/\alpha} - \epsilon_0\alpha(3/2+\alpha)H^{1+3/\alpha}. \quad (4)$$

Here, $\beta_0 \equiv P_{s0}/(B_0^2/2)$, $\gamma_0 \equiv [(X_0/r_0)^2/2]/(B_0^2/2)$, $\delta_0 \equiv [\rho_0(\Phi_0 r_0/U_0)^2/2]/(B_0^2/2)$, and $\epsilon_0 \equiv [\rho_0 G_0 M_s/(4\pi r_0)]/(B_0^2/2)$ are related, respectively, to the static thermal, poloidal-current, flow and gravitating energies normalized to the poloidal-magnetic-field energy on the reference surface. Solutions of Eq. (4) will be constructed in the low-energy regime, $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \ll 1$, and in the high-energy regime, $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1$.

3. Solution in the low-energy regime ($\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \ll 1$)

For this case it is convenient to put Eq. (4) in the form

$$\frac{d}{d\mu}\left[(1-\mu^2)^2\frac{d}{d\mu}\left(\frac{H}{1-\mu^2}\right)\right] - (1-\alpha)(2+\alpha)H = -\beta_0\alpha(2+\alpha)(1-\mu^2)H^{1+4/\alpha} -\gamma_0\alpha(1+\alpha)H^{1+2/\alpha} - \delta_0\alpha(3+\alpha)(1-\mu^2)^2H^{1+6/\alpha} - \epsilon_0(\frac{3}{2}+\alpha)(1-\mu^2)H^{1+3/\alpha}, \quad (5)$$

where $H \to 1-\mu^2$ as $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \to 0$. With the use of the boundary conditions $H(\mu \to 1) \propto 1-\mu$ and $dH/d\mu|_{\mu=0} = 0$, integration of Eq. (5) from $\mu = 0$ to $\mu = 1$ yields

$$(2+\alpha)\left[(1-\alpha)P_1 - \alpha\beta_0 P_2\right] - \alpha\left[\gamma_0(1+\alpha)P_3 + \delta_0(3+\alpha)P_4 + \epsilon_0\left(\frac{3}{2}+\alpha\right)P_5\right] = 0 \quad (6)$$

with $P_1 = \int_0^1 H d\mu$, $P_2 = \int_0^1 (1-\mu^2)H^{1+4/\alpha}$, $P_3 = \int_0^1 H^{1+2/\alpha}d\mu$ $P_4 = \int_0^1 (1-\mu^2)^2 H^{1+6/\alpha}\, d\mu$ and $P_5 = \int_0^1 (1-\mu^2)H^{1+3/\alpha}\, d\mu$. To appreciate the impact of finite pressure, finite poloidal current, flow, and gravity on the vacuum equilibrium, the relations $H = 1-\mu^2$ and $\alpha \to 1$ are employed into Eq. (6) except for the term $1-\alpha$. The departure of α from unity is then found $1-\alpha = (512/1001)\beta_0 + (16/35)\gamma_0 + (131072/230945)\delta_0 + (320/693)\epsilon_0$. Therefore, the modifications of the vacuum equilibrium from the finite pressure, poloidal current, flow, and gravity are of the order of magnitude of $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0$. As a result, an analytic solution of Eq. (5) can be derived by using the vacuum solution $H = 1-\mu^2$ in the five terms in which H appears undifferentiated. Using the boundary condition $H(\mu \to 1) \propto 1-\mu$ at $\mu = 1$ and introducing $t = 1-\mu^2 = \sin^2\theta$, integration of Eq. (5) from 1 to μ yields

$$\frac{d}{dt}\left(\frac{H}{t}\right) = \frac{1}{4t^2(1-t)^{1/2}}\left[3(1-\alpha)\int_0^t \frac{x\,dx}{(1-x)^{1/2}} - 3\beta_0\int_0^t \frac{x^6\,dx}{(1-x)^{1/2}} -2\gamma_0\int_0^t \frac{x^3\,dx}{(1-x)^{1/2}} - 4\delta_0\int_0^t \frac{x^9\,dx}{(1-x)^{1/2}} - \frac{5}{2}\epsilon_0\int_0^t \frac{x^5\,dx}{(1-x)^{1/2}}\right]. \quad (7)$$

Evaluating the integrals in Eq. (7) and integrating again, using $H(\mu = 0) = 1$, we obtain a low-energy solution, not given explicitly here, valid at all distances from a point dipole and which for vanishing poloidal current, flow and gravity ($\gamma_0 = \delta_0 = \epsilon_0 = 0$) reduces to the low-pressure static equilibrium solution of Ref. [2] [Eq. (12) therein].

4. Solution in the high-energy regime ($\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1$)

From the low-energy solution we anticipate that for $\beta_0 \approx \gamma_0 \approx \delta_0 \approx \epsilon_0 \gg 1$ it holds that $|\alpha| \ll 1$ and, consequently, assume that $1/\beta_0 \ll |\alpha| \ll 1$, to be verified *a posteriori*. The term $\alpha(1+\alpha)H/(1-\mu^2)$ in Eq. (4) is then small everywhere, and it can be neglected. Also, the terms on the RHS of Eq. (4) are large at $\mu = 0$ and rapidly decrease to zero as H decreases from $H(\mu = 0) = 1$ toward $H(\mu = 1) = 0$ since $|\alpha| \ll 1$. In particular, the variations of the terms $\gamma_0(1-\mu^2)^{-1}\alpha(1+\alpha)H^{1+2/\alpha}$ and $\delta_0(1-\mu^2)\alpha(3+\alpha)H^{1+6/\alpha}$ from $1-\mu^2$ are much weaker than those from $H^{1+2/\alpha}$ and $H^{1+6/\alpha}$, respectively. Therefore, in the above terms, $1-\mu^2$ can be approximated by unity and, consequently, Eq. (4) can be written in the form

$$\frac{d^2H}{d\mu^2} = -\alpha\left(2\beta_0 H^{1+4/\alpha} + \gamma_0 H^{1+2/\alpha} + 3\delta_0 H^{1+6/\alpha} + \frac{3}{2}\epsilon_0 H^{1+3/\alpha}\right). \tag{8}$$

Multiplying Eq. (8) by $dH/d\mu$, integrating from $\mu = 0$ where $dH/d\mu = 0$, and integrating again the resulting equation from $\mu = 0$, where $H(\mu = 0) = 1$, to μ we find

$$\begin{aligned} |\alpha|\mu \;&=\; \int_H^1 \left[\beta_0\left(1 - x^{2+4/\alpha}\right) + \gamma_0\left(1 - x^{2+2/\alpha}\right) + \delta_0\left(1 - x^{2+6/\alpha}\right)\right. \\ &\left. + \epsilon_0\left(1 - x^{2+3/\alpha}\right)\right]^{-1/2} dx \stackrel{\mu\to 1}{\longrightarrow} (1-H)(\beta_0+\gamma_0+\delta_0+\epsilon_0)^{-1/2}. \end{aligned} \tag{9}$$

To satisfy $H(\mu = 1) = 0$, Eq. (9) requires $|\alpha| = (\beta_0+\gamma_0+\delta_0+\epsilon_0)^{-1/2} + \mathcal{O}(1/(\beta_0+\gamma_0+\delta_0+\epsilon_0))$. This requirement implies that the distance between adjacent flux surfaces at the symmetry plane $\mu = 0$ increases as either of β_0, γ_0, δ_0, and ϵ_0 increases. Indeed, as α decreases the spacing must adjust to keep $U \approx (r_0/r)^\alpha$ fixed and, therefore, the magnetic surfaces become more extended and localized about the symmetry plane. The resulting equilibria resemble the accretion disks in astrophysics.

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